Chapter 10. Laser Oscillation: Gain and Threshold

Detailed description of laser oscillation

10.2 Gain

<Plane wave propagating in vacuum>
Consider a quasi-monochromatic plane wave of frequency $v$ propagating in the $+z$ direction;

![Diagram of plane wave propagating in vacuum]

The rate at which electromagnetic energy passes through a plane of cross-sectional area $A$ at $z$ is $I_v(z)A$

The flux difference:

$$[I_v(z+\Delta z) - I_v(z)]A = \frac{\partial}{\partial z}(I_v A) \Delta z$$
This difference gives the rate at which electromagnetic energy leaves the volume AΔz,

\[ \frac{\partial}{\partial t}(u_\nu A\Delta z) = \frac{\partial}{\partial z}(I_\nu A)\Delta z \]

\[ u_\nu = \frac{I_\nu}{c} \Rightarrow \frac{1}{c} \frac{\partial}{\partial t} I_\nu + \frac{\partial}{\partial z} I_\nu = 0 \] : Equation of continuity

<Plane wave propagating in medium>

The change of electromagnetic energy due to the medium should be considered.

=> The rate of change of upper(lower)-state population density due to both absorption and stimulated emission.

(7.3.4a) => energy flux added to the field: \( \sigma(v)\Phi(N_2-N_1)\cdot h\nu = \sigma(v)I_\nu(N_2-N_1) \)

\[ \left( \frac{1}{c} \frac{\partial}{\partial t} + \frac{\partial}{\partial z} \right) I_\nu = \sigma(v)I_\nu(N_2-N_1) \]
(7.4.19): \( \sigma(\nu) = (h \nu / c) BS(\nu) \)

\[
\left( \frac{1}{c} \frac{\partial}{\partial t} + \frac{\partial}{\partial z} \right) I_\nu = \frac{h \nu}{c} B (N_2 - N_1) S(\nu) I_\nu \\
= \frac{h \nu}{c} B \left( N_2 - \frac{g_2}{g_1} N_1 \right) S(\nu) I_\nu \\
= \frac{\lambda^2 A}{8\pi} \left( N_2 - \frac{g_2}{g_1} N_1 \right) S(\nu) I_\nu \\
= \sigma(\nu) \left[ N_2 - \frac{g_2}{g_1} N_1 \right] I_\nu
\]

Define, gain coefficient, \( g(\nu) \)

\[
g(\nu) = \frac{\lambda^2 A}{8\pi} \left( N_2 - \frac{g_2}{g_1} N_1 \right) S(\nu) \\
= \sigma(\nu) \left( N_2 - \frac{g_2}{g_1} N_1 \right)
\]
\[ \frac{\partial I_v}{\partial z} + \frac{1}{c} \frac{\partial I_v}{\partial t} = g(v)I_v \]

Temporal steady state: \( \frac{\partial}{\partial t} \to 0 \quad \Rightarrow \quad \frac{dI_v}{dz} = g(v)I_v \)

\[ I_v(z) = I_v(0) e^{g(v)z} \quad : \text{valid only for low intensity (saturation effect, } g = g(I)) \]

* \( g > 0 \) when \( N_2 > \frac{g_1}{g_2} N_1 \) : population inversion

* If \( N_2 \ll \frac{g_1}{g_2} N_1 \), \( N_2 \approx \frac{g_2}{g_1} N_1 \approx \frac{g_2}{g_1} N \)

\[ g(v) \approx -\frac{\lambda^2 A g_2}{8\pi g_1} N S(v) \]

\[ = -N \frac{g_2}{g_1} \sigma(v) = -a(v) \quad \text{The gain coefficient is identical, except for its sign, to the absorption coefficient.} \]
10.3 Feedback

In practice, $g \approx 0.01 \text{ cm}^{-1}$ if the length of active medium is 1 m.

$\Rightarrow$ A spontaneously emitted photon at one end of the active medium leads to a total of $e^{0.01 \times 100} = e^1 = 2.72$ photons emerging at the other end.

$\Rightarrow$ The output of such a laser is obviously not very impressive.

$\Rightarrow$ Reflective mirrors at the ends of the active medium: Feedback.

10.4 Threshold

In a laser there is not only an increase in the number of cavity photons because of stimulated emission, but also a decrease because of loss effects.

(Loss effects: scattering, absorption, diffraction, output coupling)

In order to sustain laser oscillation the stimulated amplification must be sufficient to overcome the losses.
Absorption and scattering within the gain medium is quite small compared with the loss occurring at the mirrors of the laser. => Consider only the losses associated with the mirrors.

\[ r + t + s = 1 \]

where, \( r \) : reflection coefficient
\( t \) : transmission coefficient
\( s \) : fractional loss

At the mirror at \( z=L \) and 0 :

\[ I_{v}^{(-)}(L) = r_2 I_{v}^{(+)}(L) \]

\[ I_{v}^{(+)}(0) = r_1 I_{v}^{(-)}(0) \]
In steady state (or CW operation), \( g(\nu) : \) constant

\[
\frac{dI_\nu^+}{dz} = g(\nu)I_\nu^+ \quad \frac{dI_\nu^-}{dz} = -g(\nu)I_\nu^-
\]

\[\therefore \quad I_\nu^+(z) = I_\nu^+(0)e^{g(\nu)z} \]
\[I_\nu^-(z) = I_\nu^-(L)e^{g(\nu)(L-z)} \]

Amplification condition: \( I_\nu^+(0) \) after one round trip must be higher than the initial \( I_\nu^+(0) \)

\[\therefore \quad I_\nu^+(0) = r_1I_\nu^-(0) \]
\[= r_1[e^{g(\nu)L}I_\nu^-(L)] \]
\[= r_1e^{g(\nu)L}[r_2I_\nu^+(L)] \]
\[= r_1r_2e^{g(\nu)L}[I_\nu^+(0)e^{g(\nu)L}] \]
\[= [r_1r_2e^{2g(\nu)L}]I_\nu^+(0) \geq I_\nu^+(0) \]

\[\therefore \quad r_1r_2e^{2gL} \geq 1 \]
Threshold gain, $g_t$

$$g_t = \frac{1}{2L} \ln \left( \frac{1}{r_1r_2} \right) = -\frac{1}{2L} \ln(r_1r_2)$$

* In the case of high reflectivity, $r_1r_2 \approx 1$, and $\ln(1-x) \approx -x$

$$\Rightarrow g_t = \frac{1}{2L} (1-r_1r_2) \text{ (high reflectivities, } r_1r_2 > 0.9)$$

* If the “distributed losses” (losses not associated with the mirrors) are included,

$$g_t = -\frac{1}{2L} \ln(r_1r_2) + a$$

where, $a$ : effective loss per unit length
Example) He-Ne laser, $L=50$ cm, $r_1=0.998$, $r_2=0.980$

$$g_t = -\frac{1}{2(50)} \ln[(0.998)(0.980)] \text{cm}^{-1}$$

$$= 2.2 \times 10^{-4} \text{cm}^{-1}$$

Threshold population inversion:

$$g(\nu) = \frac{\lambda^2 A}{8\pi n^2} \left( N_2 - \frac{g_2}{g_1} N_1 \right) S(\nu) \quad \Rightarrow \quad \Delta N_t = \left( N_2 - \frac{g_2}{g_1} N_1 \right) = \frac{8\pi n^2 g_t}{\lambda^2 AS(\nu)} = \frac{g_t}{\sigma(\nu)}$$

$A \approx 1.4 \times 10^6 \text{sec}^{-1}$ at $\lambda=632.8$ nm

$T \sim 400$K, $M_{Ne}=20$ amu $\Rightarrow \delta\nu_D = 2.15 \times 10^6 \left[ \frac{1}{\lambda} \left( \frac{T}{M} \right)^{1/2} \right]$ MHz $\approx 1500$ MHz

$$\Delta N_t \approx \frac{(8\pi)(2.2 \times 10^{-4} \text{cm}^{-1})}{(6328 \times 10^{-8} \text{cm})^2 (1.4 \times 10^6 \text{sec}^{-1})(6.3 \times 10^{-10} \text{sec})}$$

$$= 1.6 \times 10^9 \text{atoms/cm}^3$$

$$\frac{\Delta N_t}{N} = \frac{1.6 \times 10^9}{4.8 \times 10^{15}} = \frac{1}{3} \times 10^{-6} \quad \text{(very small !)}$$
10.5 Rate Equations for Photons and Populations

Time-dependent phenomena?

1) Gain term (stimulated emission or absorption)

\[
\frac{\partial I_v^{(\pm)}}{\partial z} + \frac{1}{c} \frac{\partial I_v^{(\pm)}}{\partial t} = g(\nu)I_v^{(\pm)}
\]

\[
-\frac{\partial I_v^{(-)}}{\partial z} + \frac{1}{c} \frac{\partial I_v^{(-)}}{\partial t} = g(\nu)I_v^{(-)}
\]

\[\Rightarrow \frac{\partial}{\partial z}\left(I_v^{(+)} - I_v^{(-)}\right) + \frac{1}{c} \frac{\partial}{\partial t}\left(I_v^{(+)} + I_v^{(-)}\right) = g(\nu)(I_v^{(+)} + I_v^{(-)})\]

In many lasers there is very little gross variation of either \(I_v^{(+)}\) or \(I_v^{(-)}\) with \(z\). \(\Rightarrow\) \(\frac{\partial}{\partial z} \rightarrow 0\)

\[
\frac{d}{dt}(I_v^{(+) + I_v^{(-)}}) = cg(\nu)(I_v^{(+) + I_v^{(-)}})
\]

\(*) l< L \) \((l : \text{gain medium length}, L : \text{cavity length})\)

\[
\frac{d}{dt}(I_v^{(+) + I_v^{(-)}}) = \frac{cl}{L} g(\nu)(I_v^{(+) + I_v^{(-)}})
\]

(10.5.4b)
2) Loss term (output coupling, absorption/scattering at the mirrors)

By the output coupling, a fraction $1-r_1 r_2$ of intensity is lost per round trip time $2L/c$.

\[
\frac{d}{dt}(I_v^{(+) + I_v^{(-})}) = -\frac{c}{2L}(1-r_1 r_2)(I_v^{(+) + I_v^{(-})})
\]  

(10.5.8)

From (10.5.4b), (10.5.8), and put $I_v = I_v^{(+) + I_v^{(-})} : \text{total intensity}$

\[
\frac{dI_v}{dt} = \frac{c l}{L} g(v) I_v - \frac{c}{2L} (1-r_1 r_2) I_v
\]

or photon number, $q_v \propto I_v$

\[
\frac{dq_v}{dt} = \frac{c l}{L} g(v) q_v - \frac{c}{2L} (1-r_1 r_2) q_v
\]

\[= \frac{c l}{L} g(v) q_v - \frac{c l}{L} g_r q_v\]
If $g_1 = g_2$, 
\[
\frac{dI_v}{dt} = \frac{cl}{L} \frac{\lambda^2 A}{8\pi} (N_2 - N_1)S(\nu)I_v - \frac{c}{2L} (1 - r_1r_2)I_v \\
= \frac{cl}{L} \sigma(\nu) (N_2 - N_1)I_v - \frac{c}{2L} (1 - r_1r_2)I_v
\]

Coupled equations for the light and the atoms in the laser cavity including pumping effect:

\[
\frac{dN_1}{dt} = -\Gamma_1 N_1 + AN_2 + g(\nu)\Phi_v \\
\frac{dN_2}{dt} = - (\Gamma_2 + A) N_2 - g(\nu)\Phi_v + K \\
\frac{d\Phi_v}{dt} = \frac{cl}{L} g(\nu)\Phi_v - \frac{c}{2L} (1 - r_1r_2)\Phi_v
\]
10.7 Three-Level Laser Scheme

1) Two-level laser scheme is not possible.

Neglecting $\Gamma_1$, $\Gamma_2$ in (7.3.2) => $N_2(\infty) = \frac{N\sigma\Phi}{A_{21} + 2\sigma\Phi} < \frac{N}{2}$ : can’t achieve the population inversion

2) Three-level laser scheme

\[ \begin{align*}
\frac{dN_1}{dt} &= -PN_1 + \Gamma_{21}N_2 + \sigma\Phi_v (N_2 - N_1) \\
\frac{dN_2}{dt} &= PN_1 - \Gamma_{21}N_2 - \sigma\Phi_v (N_2 - N_1) \\
\frac{d\Phi_v}{dt} &= \frac{cl}{L} g(\nu) \Phi_v - \frac{c}{2L} (1-r_1r_2) \Phi_v
\end{align*} \]

P : pumping rate

\[ \left( \frac{dN_1}{dt} \right)_{\text{pumping}} = -PN_1 \]

\[ \left( \frac{dN_2}{dt} \right)_{\text{pumping}} \approx \left( \frac{dN_3}{dt} \right)_{\text{pumping}} = -\left( \frac{dN_1}{dt} \right)_{\text{pumping}} = PN_1 \]
<Threshold pumping> for the steady-state operation

i) Near threshold the number of cavity photons is small enough that stimulated
emission may be omitted from Eq. (10.7.4)

ii) Steady-state: \( \dot{N}_1 = \dot{N}_2 = 0 \)

\[ \Rightarrow \quad \dot{N}_2 = \frac{P}{\Gamma_{21}} \dot{N}_1 \]

And, \( \bar{N}_1 + \bar{N}_2 = N_T \) : total population is conserved

\[ \Rightarrow \quad \bar{N}_1 = \frac{\Gamma_{21}}{P + \Gamma_{21}} N_T \quad \bar{N}_2 = \frac{P}{P + \Gamma_{21}} N_T \quad \bar{N}_2 - \bar{N}_1 = \frac{P - \Gamma_{21}}{P + \Gamma_{21}} N_T \]

# Positive (steady-state) population inversion: \( P > \Gamma_{21} \)

\( P_{\text{min}} = \Gamma_{21} \)

# Threshold pumping power:

\[ \frac{P_{\text{WR}}}{V} = h \nu_{31} \bar{P} \bar{N}_1 = \frac{h \nu_{31} P \Gamma_{21}}{P + \Gamma_{21}} N_T = \frac{1}{2} \Gamma_{21} N_T h \nu_{31} \]
10.8 Four-Level Laser Scheme

Lower laser level is not the ground level: The depletion of the lower laser level obviously enhances the population inversion.

Population rate equations:

\[
\frac{dN_0}{dt} = -PN_0 + \Gamma_{10} N_1 \\
\frac{dN_1}{dt} = -\Gamma_{10} N_1 + \Gamma_{21} N_2 + \sigma(\nu)(N_2 - N_1)\Phi_\nu \\
\frac{dN_2}{dt} = PN_0 - \Gamma_{21} N_2 - \sigma(\nu)(N_2 - N_1)\Phi_\nu
\]
\begin{align*}
N_0 + N_1 + N_2 &= \text{constant} = N_T \\
\text{Steady-state solutions} : \\
\bar{N}_0 &= \frac{\Gamma_{10}\Gamma_{21}}{\Gamma_{10}\Gamma_{21} + \Gamma_{10}P + \Gamma_{21}P} N_T \\
\bar{N}_1 &= \frac{\Gamma_{21}P}{\Gamma_{10}\Gamma_{21} + \Gamma_{10}P + \Gamma_{21}P} N_T \\
\bar{N}_2 &= \frac{\Gamma_{10}P}{\Gamma_{10}\Gamma_{21} + \Gamma_{10}P + \Gamma_{21}P} N_T \\
\text{If } \Gamma_{10} \gg \Gamma_{21}, P \\
\bar{N}_2 - \bar{N}_1 &\approx \bar{N}_2 \approx \frac{P}{P + \Gamma_{21}} N_T \\
\text{Population inversion : } \\
\bar{N}_2 - \bar{N}_1 &= \frac{P(\Gamma_{10} - \Gamma_{21})N_T}{\Gamma_{10}\Gamma_{21} + \Gamma_{10}P + \Gamma_{21}P} \\
\# \text{ Positive inversion : } \Gamma_{10} > \Gamma_{21} \\
&\text{lower level decays more rapidly than the upper level.}
\end{align*}
10.9 Comparison of Pumping Requirements for Three and Four-Level Lasers

1) Threshold pumping rate, $P_t$

\[(10.7.9) \Rightarrow (P_t)_{\text{three-level laser}} = \frac{N_T + \Delta N_t}{N_T - \Delta N_t} \Gamma_{21}\]

\[(10.8.7), (10.8.6) \Rightarrow (P_t)_{\text{four-level laser}} = \frac{\Delta N_t}{N_T - \Delta N_t} \Gamma_{21}\]

\[\therefore \frac{(P_t)_{\text{four-level laser}}}{(P_t)_{\text{three-level laser}}} = \frac{\Delta N_t}{N_T - \Delta N_t} << 1\]

2) Threshold pumping power, $(Pwr)_t$

\[(10.7.15) \Rightarrow \frac{(Pwr)_t}{V}_{\text{three-level laser}} = \frac{1}{2} h \nu_{31} N_T \Gamma_{21}\]

\[\therefore\]

\[\frac{(Pwr)_t}{V}_{\text{four-level laser}} \approx h \nu_{30} \Delta N_t \Gamma_{21}\]

\[\therefore\]

\[\frac{((Pwr)_t/V)_{\text{four-level laser}}}{((Pwr)_t/V)_{\text{three-level laser}}} \approx \frac{2 \nu_{30} \Delta N_t}{\nu_{31} N_T}\]
10.11 Small-Signal Gain and Saturation

For the three-level laser scheme, steady-state solution including the stimulated emission term:

\[ \bar{N}_2 - \bar{N}_1 = \frac{(P - \Gamma_{21})N_T}{P + \Gamma_{21} + 2\sigma \Phi_v} \]

\[ g(\nu) = \frac{\sigma(\nu)(P - \Gamma_{21})N_T}{P + \Gamma_{21} + 2\sigma(\nu)\Phi_v} = \text{ln}(\Phi_{\nu}^{-1}) \]

\[ = A \text{ large photon number, and therefore a large stimulated emission rate, tends to equalize the populations } \bar{N}_1 \text{ and } \bar{N}_2 \text{. In this case, the gain is said to be “saturated.”} \]

<Microscopic view of the gain saturation>

As the cavity photon number increased, the stimulated absorption as well as the stimulated emission increased. \( \Rightarrow \) The lower level’s absorption rate is exactly equal to the upper level’s emission rate in the extreme limit. \( \Leftrightarrow \) The gain is zero.
<Small signal gain / Saturation flux>

\[ g(\nu) = \frac{\sigma(\nu)(P - \Gamma_{21})N_T}{P + \Gamma_{21}} \frac{1}{1 + [2\sigma(\nu)\Phi_\nu / (P + \Gamma_{21})]} \]

\[ = \frac{g_0(\nu)}{1 + \Phi_\nu / \Phi_\nu^{\text{sat}}} \]

Where we define, Small signal gain as

\[ g_0(\nu) = \frac{\sigma(\nu)(P - \Gamma_{21})N_T}{P + \Gamma_{21}} \]

Saturation flux as

\[ \Phi_\nu^{\text{sat}} = \frac{P + \Gamma_{21}}{2\sigma(\nu)} \]

* The larger the decay rates, the larger the saturation flux.
* The saturation flux \( \Leftrightarrow \) Stimulated emission rate:
  
  The average of the upper- and lower-level decay rates.
<Gain width>

When the absorption lineshape is Lorentzian,

\[ \sigma(\nu) = \frac{1}{(\nu_{21} - \nu)^2/(\delta\nu_{21})^2 + 1} \]

* Small signal gain width: \( \Delta\nu_g = \delta\nu_{21} \)

(10.11.3) =>

\[ g(\nu) = g_0(\nu_{21}) \frac{1}{(\nu_{21} - \nu)^2/(\delta\nu_{21})^2 + 1 + (\Phi_\nu / \Phi_{\nu_{21}}^{sat})} \]

where, \( g_0(\nu_{21}) = \frac{\sigma(\nu_{21})(P - \Gamma_{21})N_T}{P + \Gamma_{21}} \) : Line-center small signal gain

* Power-broadened gain width:

\[ \Delta\nu_g = \delta\nu_{21} \left( 1 + \frac{\Phi_\nu}{\Phi_{\nu_{21}}^{sat}} \right)^{1/2} \]
\[ \Phi_{\nu_21}^{\text{sat}} = \frac{P + \Gamma_{21}}{2\sigma(\nu_{21})} = \frac{4\pi^2 \delta \nu_{21}}{\lambda^2 A} (P + \Gamma_{21}) \propto \delta \nu_{21} \]

Line-center saturation flux is directly proportional to the transition linewidth.
10.12 Spatial Hole Burning

In most laser, we have standing waves rather than traveling waves.

=> Cavity standing wave field is the sum of two oppositely propagating traveling wave fields:

\[ E(z,t) = E_0 \cos \omega t \sin k_z \]

\[ = \frac{1}{2} E_0 [\sin (k_z - \omega t) + \sin (k_z + \omega t)] \]

\[ = E_+ (z,t) + E_- (z,t) \]

where, \( E_\pm (z,t) = \frac{1}{2} E_0 \sin (k_z \mp \omega t) \)

The time-averaged square of the electric field gives a field energy density:

\[ \frac{h \nu}{c} \Phi_\nu^{(\pm)} = \frac{\varepsilon_0}{8} E_0^2 \]

=> Total photon flux,

\[ \Phi_\nu = 2[\Phi_\nu^{(+) + \Phi_\nu^{(-)}} \sin^2 k_z \]

\[ (I_\nu = h \nu \Phi_\nu) \Rightarrow I_\nu = 2[I_\nu^{(+) + I_\nu^{(-)}} \sin^2 k_z \]

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\[ g(\nu) = \frac{g_0(\nu)}{1 + 2\left[\left(\Phi^{(+)}_\nu + \Phi^{(-)}_\nu\right) / \Phi^{\text{sat}}_\nu\right] \sin^2 k_z} \]