Dispersion relation of metal nanorod and nanotip

D. E. Chang, A. S. Sørensen, P. R. Hemmer, and M. D. Lukin, “*Strong coupling of single emitters to surface plasmons*,” PR B 76,035420 (2007)

M. I. Stockman, “*Nanofocusing of Optical Energy in Tapered Plasmonic Waveguides*,” Phys. Rev. Lett. 93, 137404 (2004)

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Guiding of a one-dimensional optical beam with nanometer diameter

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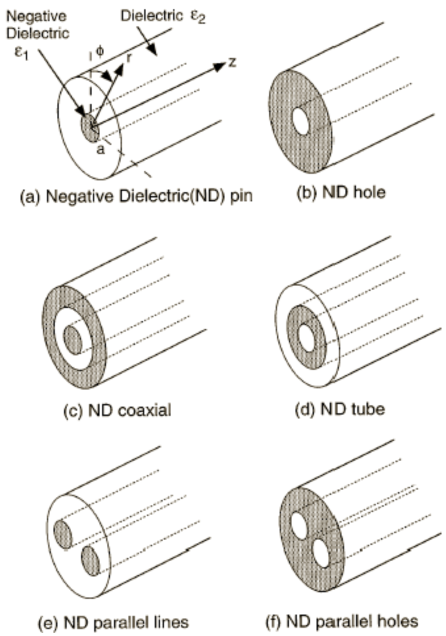
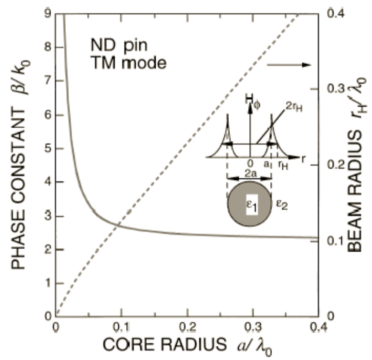
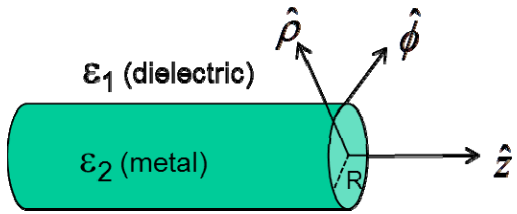
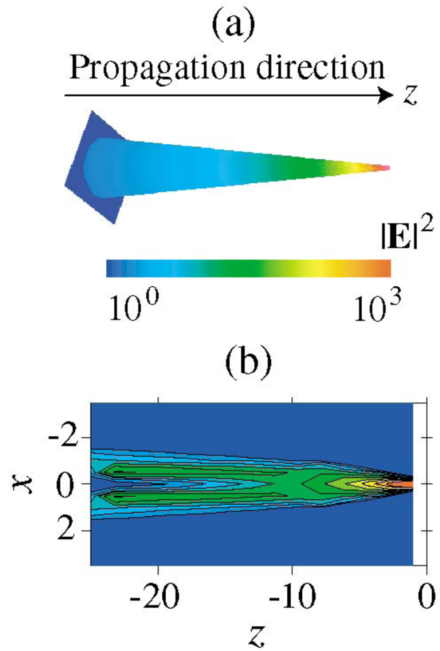


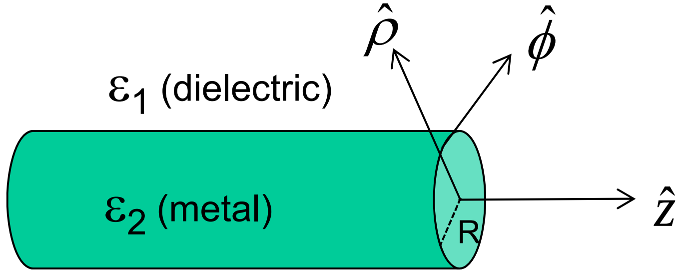






A. Dispersion relation of metal nanorods

D. E. Chang, A. S. Sørensen, P. R. Hemmer, and M. D. Lukin,
"Strong coupling of single emitters to surface plasmons," PR B 76,035420 (2007)



For nonmagnetic media, the electric and magnetic fields in frequency space satisfy the wave equation,

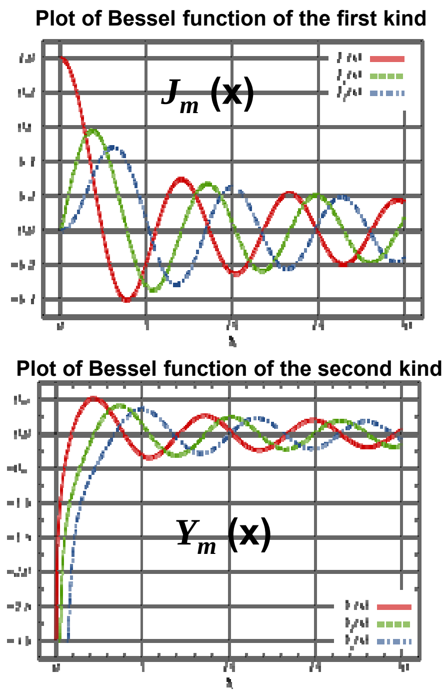
$$\nabla^2 \begin{Bmatrix} \mathbf{E}(\mathbf{r}) \\ \mathbf{H}(\mathbf{r}) \end{Bmatrix} + \frac{\omega^2}{c^2}\epsilon(\mathbf{r}) \begin{Bmatrix} \mathbf{E}(\mathbf{r}) \\ \mathbf{H}(\mathbf{r}) \end{Bmatrix} = 0$$

In cylindrical coordinates, the electric field is given by

$$\mathbf{E}_i(\mathbf{r}) = \mathcal{E}_{i,m}\mathbf{E}_{i,m}(k_{i\perp}\rho)e^{im\phi}e^{ik_{\parallel}z}$$

$$k_{i\perp} = \sqrt{k_i^2 - k_{\parallel}^2}$$

$$k_i = \omega\sqrt{\epsilon_i}/c$$



The scalar solutions of the wave equations satisfying the necessary boundary conditions take the form,

(outside: $\rho > R$) $\quad \psi_1 \propto H_m(k_{1\perp}\rho)e^{im\phi + ik_{\parallel}z}$

(inside : $\rho < R$) $\quad \psi_2 \propto J_m(k_{2\perp}\rho)e^{im\phi + ik_{\parallel}z}$

J_m : Bessel functions of the first kind

H_m : Hankel functions of the first kind

$$H_m(x) = J_m(x) + iY_m(x)$$

NOTE : Bessel functions and Hankel functions

Bessel's Differential Equation is defined as: $$x^2y''+xy'+\left(x^2-n^2\right)y=0 \qquad n=0,1,2,3,\ldots$$

The solutions of this equation are called **Bessel Functions** of order n.

Type	First kind	Second kind
Bessel functions	J_α	Y_α
Modified Bessel functions	I_α	K_α
Hankel functions	$H_\alpha^{(1)}=J_\alpha+iY_\alpha$	$H_\alpha^{(2)}=J_\alpha-iY_\alpha$
Spherical Bessel functions	j_n	y_n
Spherical Hankel functions	$h_n^{(1)}=j_n+iy_n$	$h_n^{(2)}=j_n-iy_n$

Two sets of functions:
the Bessel function of the first kind ➔ $J_n(x)$
the Bessel function of the second kind (also known as the Weber Function) ➔ $Y_n(x)$

$$y(x)=c_1J_n(x)+c_2Y_n(x)$$

Bessel 1st and 2nd Functions:

$$J_n(x)=\frac{x^n}{2^n\Gamma(n+1)}\left[1-\frac{x^2}{2(2n+2)}+\frac{x^4}{2\cdot4(2n+2)(2n+4)}-\cdots\right]=\sum_{k=0}^{\infty}\frac{(-1)^k(x/2)^{n+2k}}{k!\,\Gamma(n+k+1)}$$

$$Y_n(x)=\frac{2}{\pi}J_n(x)\left(\ln\frac{x}{2}+\gamma\right)-\frac{1}{\pi}\sum_{m=0}^{n-1}\frac{(n-m-1)!}{m!}\left(\frac{x}{2}\right)^{2m-n}+\frac{1}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^{m-1}\left[\left(1-\frac{1}{2}+\ldots+\frac{1}{m}\right)+\left(1+\frac{1}{2}+\ldots+\frac{1}{m+n}\right)\right]}{m!(m-n)!}\left(\frac{x}{2}\right)^{2m+n}$$
$$=\lim_{p\to n}\frac{J_p(x)\cos p\pi-J_{-p}(x)}{\sin p\pi} \qquad n=0,1,2,\ldots$$

For small x, $x\to0$

$$J_n\sim\frac{1}{2^n n!}x^n \qquad Y_0\sim\frac{2}{\pi}\ln x \qquad Y_n\sim-\frac{2^n(n-1)!}{\pi}x^{-n}$$

For large x, $x\gg n$

$$J_n(x)\sim\sqrt{\frac{2}{\pi x}}\cos\left[x-(2n+1)\frac{\pi}{4}\right] \qquad Y_n(x)\sim\sqrt{\frac{2}{\pi x}}\sin\left[x-(2n+1)\frac{\pi}{4}\right]$$

Recurrence Relation:

$$J_n'(x)=\frac{1}{2}\left[J_{n-1}(x)-J_{n+1}(x)\right] \qquad Y_n'(x)=\frac{1}{2}\left[Y_{n-1}(x)-Y_{n+1}(x)\right]$$

Hankel Function:

the Hankel function of the first kind and second kind, prominent in the theory of wave propagation, are defined as

$$H_\nu^{(1)}(x)=J_\nu(x)+iY_\nu(x) \qquad H_\nu^{(2)}(x)=J_\nu(x)-iY_\nu(x)$$

For large x, $x\gg n$

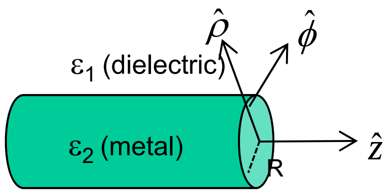
$$H_n^{(1)}(x)\sim\sqrt{\frac{2}{\pi x}}e^{i\left(x-\frac{\pi}{4}-\frac{n\pi}{2}\right)} \qquad H_n^{(2)}(x)\sim\sqrt{\frac{2}{\pi x}}e^{-i\left(x-\frac{\pi}{4}-\frac{n\pi}{2}\right)}$$

Modified Bessel Function:

$$I_n(x)=i^{-n}J_n(ix)$$

$$K_n(x)=\frac{\pi}{2}i^{n+1}\left[J_n(ix)+iY_n(ix)\right]=\frac{\pi}{2}i^{n+1}H_n^{(1)}(ix)$$

For nonmagnetic media, the electric and magnetic fields in frequency space satisfy the wave equation,



$$\nabla^2 \begin{Bmatrix} \mathbf{E}(\mathbf{r}) \\ \mathbf{H}(\mathbf{r}) \end{Bmatrix} + \frac{\omega^2}{c^2}\epsilon(\mathbf{r}) \begin{Bmatrix} \mathbf{E}(\mathbf{r}) \\ \mathbf{H}(\mathbf{r}) \end{Bmatrix} = 0$$

The scalar solutions of the wave equations satisfying the necessary boundary conditions take the form,

(outside: $\rho > R$) $\psi_1 \propto H_m(k_{1\perp}\rho)e^{im\phi + ik_\parallel z}$

(inside : $\rho < R$) $\psi_2 \propto J_m(k_{2\perp}\rho)e^{im\phi + ik_\parallel z}$

$$k_{i\perp} = \sqrt{k_i^2 - k_\parallel^2} \qquad k_i = \omega\sqrt{\epsilon_i}/c$$

$\mathbf{v}_i = \frac{1}{k_i}\nabla \times (\hat{z}\psi_i)$ and $\mathbf{w}_i = \frac{1}{k_i}\nabla \times \mathbf{v}_i$. $\mathbf{E}_i(\mathbf{r}) = a_i\mathbf{v}_i(\mathbf{r}) + b_i\mathbf{w}_i(\mathbf{r})$, $\mathbf{H}_i(\mathbf{r}) = -\dfrac{i}{\omega\mu_0}k_i[a_i\mathbf{w}_i(\mathbf{r}) + b_i\mathbf{v}_i(\mathbf{r})]$, where a_i and b_i are constant coefficients.

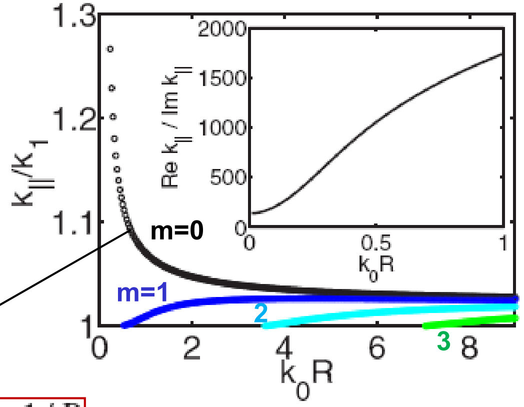
$$\mathbf{E}_i(\mathbf{r}) = \left\{ \left[\frac{im}{k_i\rho}a_iF_{i,m}(k_{i\perp}\rho) + \frac{ik_\parallel k_{i\perp}}{k_i^2}b_iF'_{i,m}(k_{i\perp}\rho) \right]\hat{\rho} + \left[-\frac{k_{i\perp}}{k_i}a_iF'_{i,m}(k_{i\perp}\rho) - \frac{mk_\parallel}{k_i^2\rho}b_iF_{i,m}(k_{i\perp}\rho) \right]\hat{\phi} + \frac{k_{i\perp}^2}{k_i^2}b_iF_{i,m}(k_{i\perp}\rho)\hat{z} \right\} e^{im\phi + ik_\parallel z}$$

$$\mathbf{H}_i(\mathbf{r}) = -\frac{i}{\omega\mu_0}k_i \left\{ \left[\frac{ik_\parallel k_{i\perp}}{k_i^2}a_iF'_{i,m}(k_{i\perp}\rho) + \frac{im}{k_i\rho}b_iF_{i,m}(k_{i\perp}\rho) \right]\hat{\rho} - \left[\frac{mk_\parallel}{k_i^2\rho}a_iF_{i,m}(k_{i\perp}\rho) + \frac{k_{i\perp}}{k_i}b_iF'_{i,m}(k_{i\perp}\rho) \right]\hat{\phi} + \frac{k_{i\perp}^2}{k_i^2}a_iF_{i,m}(k_{i\perp}\rho)\hat{z} \right\} e^{im\phi + ik_\parallel z}$$

where $F_{1,m}(x) = H_m(x)$ and $F_{2,m}(x) = J_m(x)$.

(silver nanorod)

$\lambda_0 = 1\ \mu\text{m}$ $\epsilon_1 = 2$ $\epsilon_2 \approx -50 + 0.6i$



Continuity of the tangential field components *at* $\rho = R$ gives the **dispersion relation**,

$$\frac{m^2k_\parallel^2}{R^2}\left(\frac{1}{k_{2\perp}^2} - \frac{1}{k_{1\perp}^2}\right)^2 = \left[\frac{1}{k_{2\perp}}\frac{J'_m(k_{2\perp}R)}{J_m(k_{2\perp}R)} - \frac{1}{k_{1\perp}}\frac{H'_m(k_{1\perp}R)}{H_m(k_{1\perp}R)}\right] \times \left[\frac{k_2^2}{k_{2\perp}}\frac{J'_m(k_{2\perp}R)}{J_m(k_{2\perp}R)} - \frac{k_1^2}{k_{1\perp}}\frac{H'_m(k_{1\perp}R)}{H_m(k_{1\perp}R)}\right]$$

- $k_\parallel > k_1$, $\longrightarrow$ $k_{i\perp} = \sqrt{k_i^2 - k_\parallel^2} \equiv i\kappa_{i\perp}$: purely imaginary
- All higher-order modes $(|m| \geq 1)$ *exhibit a cutoff as* $R \to 0$
- The *m=0 fundamental plasmon mode* exhibits a unique behavior of $k_\parallel \propto 1/R$ $\kappa_{1\perp} \propto 1/R$
- The m=0 field outside the wire becomes tightly localized on a scale of *R around the* metal surface, leading to a small effective transverse mode area that scales like $A_{\text{eff}} \propto R^2$ →confined well below the diffraction limit!

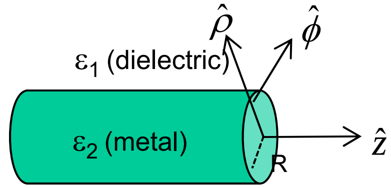
For the special case a TM mode ($H_z = 0$) with no winding $m=0$ (fundamental mode).

$$\mathbf{E}_i(\mathbf{r}) = \left\{ \left[\frac{im}{k_i\rho} a_i F_{i,m}(k_{i\perp}\rho) + \frac{ik_\parallel k_{i\perp}}{k_i^2} b_i F'_{i,m}(k_{i\perp}\rho) \right]\hat{\rho} + \left[-\frac{k_{i\perp}}{k_i} a_i F'_{i,m}(k_{i\perp}\rho) - \frac{mk_\parallel}{k_i^2\rho} b_i F_{i,m}(k_{i\perp}\rho) \right]\hat{\phi} + \frac{k_{i\perp}^2}{k_i^2} b_i F_{i,m}(k_{i\perp}\rho)\hat{z} \right\} e^{im\phi+ik_\parallel z}$$

$$\mathbf{H}_i(\mathbf{r}) = -\frac{i}{\omega\mu_0} k_i \left\{ \left[\frac{ik_\parallel k_{i\perp}}{k_i^2} a_i F'_{i,m}(k_{i\perp}\rho) + \frac{im}{k_i\rho} b_i F_{i,m}(k_{i\perp}\rho) \right]\hat{\rho} - \left[\frac{mk_\parallel}{k_i^2\rho} a_i F_{i,m}(k_{i\perp}\rho) + \frac{k_{i\perp}}{k_i} b_i F'_{i,m}(k_{i\perp}\rho) \right]\hat{\phi} + \frac{k_{i\perp}^2}{k_i^2} a_i F_{i,m}(k_{i\perp}\rho)\hat{z} \right\} e^{im\phi+ik_\parallel z}$$

where $F_{1,m}(x) = H_m(x)$ and $F_{2,m}(x) = J_m(x)$.

(TM mode, $H_z = 0$, with $m = 0$ ➔ $a_i = 0$ ➔ $E_\phi = 0$)



Continuity of the remaining tangential field components E_z *and* H_ϕ *at the* boundary requires that

$$\begin{pmatrix} \frac{k_{1\perp}^2}{k_1^2} H_0(k_{1\perp}R) & -\frac{k_{2\perp}^2}{k_2^2} J_0(k_{2\perp}R) \\ \frac{i}{\omega\mu_0} k_{1\perp} H_0'(k_{1\perp}R) & -\frac{i}{\omega\mu_0} k_{2\perp} J_0'(k_{2\perp}R) \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

(outside: $\rho > R$) $\psi_1 \propto H_m(k_{1\perp}\rho) e^{im\phi+ik_\parallel z}$

(inside : $\rho < R$) $\psi_2 \propto J_m(k_{2\perp}\rho) e^{im\phi+ik_\parallel z}$

Setting the determinant of the above matrix equal to zero (det *M=0*) immediately yields the dispersion relation,

$$\frac{k_2^2}{k_{2\perp}} \frac{J_0'(k_{2\perp}R)}{J_0(k_{2\perp}R)} - \frac{k_1^2}{k_{1\perp}} \frac{H_0'(k_{1\perp}R)}{H_0(k_{1\perp}R)} = 0 \longrightarrow \frac{b_1}{b_2} = \frac{k_{2\perp}}{k_{1\perp}} \frac{J_0'(k_{2\perp}R)}{H_0'(k_{1\perp}R)}$$

In the limit of $k_{i\perp} = \sqrt{k_i^2 - k_\parallel^2} \approx ik_\parallel$ $\longrightarrow$ $\frac{\epsilon_2}{\epsilon_1} = \frac{K_0'(k_\parallel R) I_0(k_\parallel R)}{K_0(k_\parallel R) I_0'(k_\parallel R)}$ where I_m, K_m *are modified Bessel functions*

When $|k_\parallel R| \ll 1$ (corresponding to large $|\epsilon_2/\epsilon_1|$), $\longrightarrow$ $\frac{\epsilon_2}{\epsilon_1} = \frac{2}{(\gamma - \log 2 + \log C)(C)^2}$, $k_\parallel R = C$

(nanoscale-radius wire)

$\gamma \approx 0.577$ is Euler's constant.

The fields themselves are given by

$$\mathbf{E}_1 = b_1 \left[\frac{ik_\parallel k_{1\perp}}{k_1^2} H_0'(k_{1\perp}\rho)\hat{\rho} + \frac{k_{1\perp}{}^2}{k_1^2} H_0(k_{1\perp}\rho)\hat{z} \right] e^{ik_\parallel z} \qquad \mathbf{H}_1 = \frac{i}{\omega\mu_0} k_{1\perp} b_1 H_0'(k_{1\perp}\rho) e^{ik_\parallel z}\hat{\phi}$$

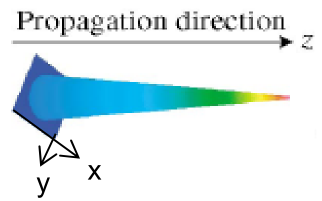
$$\mathbf{E}_2 = b_2 \left[\frac{ik_\parallel k_{2\perp}}{k_2^2} J_0'(k_{2\perp}\rho)\hat{\rho} + \frac{k_{2\perp}{}^2}{k_2^2} J_0(k_{2\perp}\rho)\hat{z} \right] e^{ik_\parallel z} \qquad \mathbf{H}_2 = \frac{i}{\omega\mu_0} k_{2\perp} b_2 J_0'(k_{2\perp}\rho) e^{ik_\parallel z}\hat{\phi}$$

$k_{i\perp} = \sqrt{k_i^2 - k_\parallel^2}$

$k_i = \omega\sqrt{\epsilon_i}/c$

B. Dispersion relation of metal nanotips

M. I. Stockman, “Nanofocusing of Optical Energy in Tapered Plasmonic Waveguides,”
Phys. Rev. Lett. 93, 137404 (2004)



Note that the *TM, fundamental mode* ($E_\phi = 0$, $H_z = 0$) *on a nanorod* was given by

$$\mathbf{E}_1 = b_1\left[\frac{ik_\parallel k_{1\perp}}{k_1^2}H_0'(k_{1\perp}\rho)\hat{\rho} + \frac{{k_{1\perp}}^2}{k_1^2}H_0(k_{1\perp}\rho)\hat{z}\right]e^{ik_\parallel z} \qquad \mathbf{E}_2 = b_2\left[\frac{ik_\parallel k_{2\perp}}{k_2^2}J_0'(k_{2\perp}\rho)\hat{\rho} + \frac{{k_{2\perp}}^2}{k_2^2}J_0(k_{2\perp}\rho)\hat{z}\right]e^{ik_\parallel z}$$

In the eikonal (WKB) approximation (slowly varying in z direction), this field *on a nanotip may have* the form

$$\mathbf{E}\,(\mathbf{r}, z, t) = \mathbf{E}_0(\mathbf{r})A(z)\exp[ik_0\varphi(\mathbf{r}) - i\omega t]$$

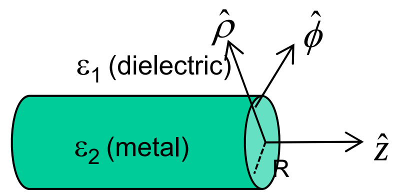
where ***r*** is a two-dimensional (2D) vector in the *xy plane and **A(z)** is a slow-varying preexponential* factor.

$k_0\varphi(\mathbf{r})$ → In the limit of $k_{i\perp} = \sqrt{k_i^2 - k_\parallel^2} \approx ik_\parallel$ (eikonal approximation?) → $\varphi = k_0 \int n(z)dz,$

where *n(z) is the effective surface index* of the plasmonic waveguide at a point *z, which is determined by the equation*

The dispersion relation obtained from the boundary conditions is,

$$\frac{k_2^2}{k_{2\perp}}\frac{J_0'(k_{2\perp}R)}{J_0(k_{2\perp}R)} - \frac{k_1^2}{k_{1\perp}}\frac{H_0'(k_{1\perp}R)}{H_0(k_{1\perp}R)} = 0 \;\rightarrow\; \text{In the limit of } k_{i\perp} = \sqrt{k_i^2 - k_\parallel^2} \approx ik_\parallel \;\rightarrow\; \frac{\epsilon_2}{\epsilon_1} = \frac{K_0'(k_\parallel R)I_0(k_\parallel R)}{K_0(k_\parallel R)I_0'(k_\parallel R)}$$



$$I_n'(x) = \frac{n}{x}I_n(x) + I_{n+1}(x)$$
$$K_n'(x) = \frac{n}{x}K_n(x) - K_{n+1}(x)$$

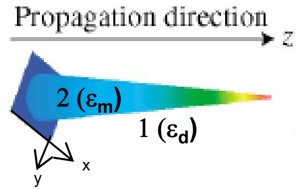
$$\frac{\varepsilon_m}{\kappa_m}\frac{I_1(k_0\kappa_m R)}{I_0(k_0\kappa_m R)} + \frac{\varepsilon_d}{\kappa_d}\frac{K_1(k_0\kappa_d R)}{K_0(k_0\kappa_d R)} = 0$$

$k_\parallel = n(z)k_0$ $\qquad \kappa_m = \sqrt{n^2 - \varepsilon_m}$ $\qquad \varepsilon_m = \varepsilon_2$ (metal)

$\kappa_d = \sqrt{n^2 - \varepsilon_d}$ $\qquad \varepsilon_d = \varepsilon_1$ (dielectric)

The SPP electric fields are found from the Maxwell equations in eikonal (WKB) approximation in the form:

For a nanorod $\left[R(z)=R;\ \text{constant}\right]$



$$\mathbf{E}_1 = b_1\left[\frac{ik_\parallel k_{1\perp}}{k_1^2}H_0'(k_{1\perp}\rho)\hat{\rho}+\frac{k_{1\perp}{}^2}{k_1^2}H_0(k_{1\perp}\rho)\hat{z}\right]e^{ik_\parallel z}$$

$$\mathbf{E}_2 = b_2\left[\frac{ik_\parallel k_{2\perp}}{k_2^2}J_0'(k_{2\perp}\rho)\hat{\rho}+\frac{k_{2\perp}{}^2}{k_2^2}J_0(k_{2\perp}\rho)\hat{z}\right]e^{ik_\parallel z}$$

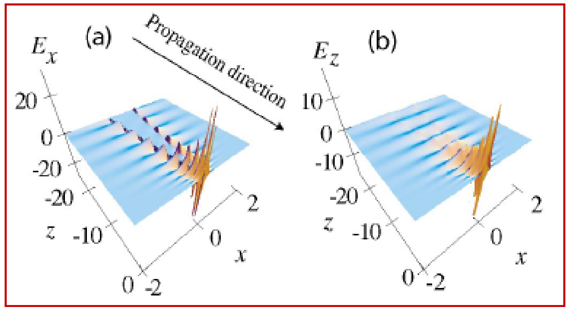
$$\frac{b_1}{b_2}=\frac{k_{2\perp}}{k_{1\perp}}\frac{J_0'(k_{2\perp}R)}{H_0'(k_{1\perp}R)}$$

For a nanotip $\left[R=R(z);\ \text{not fixed}\right]$

$$\mathbf{E}\,(\mathbf{r},z,t)=\mathbf{E}_0(\mathbf{r})A(z)\exp[ik_0\varphi(\mathbf{r})-i\omega t] \qquad \varphi = k_0\int n(z)dz,$$

$$E_1(r>R,z)=A(z)\frac{I_0(k_0\kappa_m R)}{K_0(k_0\kappa_m R)}\left[i\frac{n}{\kappa_d}K_1(k_0\kappa_d r)\hat{\rho}+K_0(k_0\kappa_d r)\hat{z}\right]e^{ink_0 z}$$

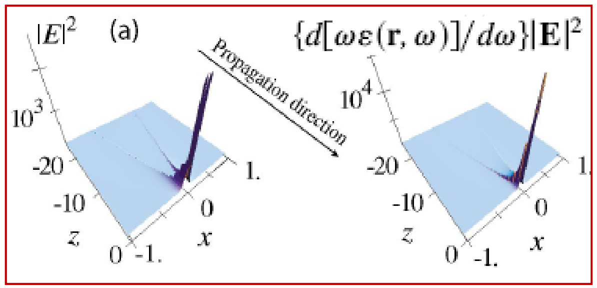
$$E_2(r<R,z)=A(z)\left[i\frac{n}{\kappa_m}I_1(k_0\kappa_m r)\hat{\rho}+I_0(k_0\kappa_m r)\hat{z}\right]e^{ink_0 z}$$



To determine the preexponential *A(z),*
we use the energy flux conservation in terms of
the Pointing vector integrated over the normal *(xy) plane,*

Intensity

Energy density



$$\Phi_{em}=\oint_S \mathbf{S}\cdot d\mathbf{a}=\oint_S(\mathbf{E}\times\mathbf{H})\cdot d\mathbf{a}$$

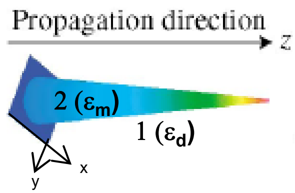
$$\longrightarrow\ A\propto \mathrm{Re}\left[\frac{n^*\varepsilon_m^*}{|\kappa_m|^2}\left|K_0(k_0\kappa_d R)\right|^2\int_0^R\left|I_1(k_0\kappa_m r)\right|^2 rdr + \frac{n^*\varepsilon_d^*}{|\kappa_d|^2}\left|I_0(k_0\kappa_m R)\right|^2\int_R^\infty\left|K_1(k_0\kappa_d r)\right|^2 rdr\right]^{-\frac{1}{2}}$$

The SPP electric fields are found from the Maxwell equations in eikonal (WKB) approximation in the form:

For a nanotip $[R = R(z);\ \text{not fixed}]$ $\quad \mathbf{E}(\mathbf{r}, z, t) = \mathbf{E}_0(\mathbf{r}) A(z) \exp[ik_0\varphi(\mathbf{r}) - i\omega t] \quad \varphi = k_0 \int n(z)dz,$

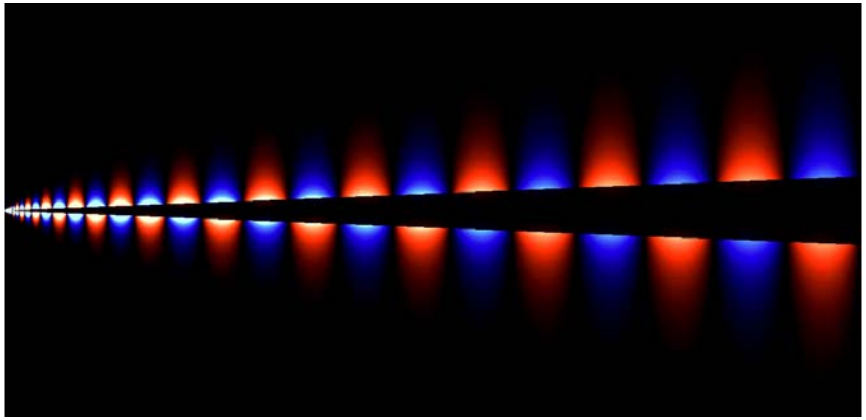
$$E_1(r > R, z) = A(z)\frac{I_0(k_0\kappa_m R)}{K_0(k_0\kappa_m R)}\left[i\frac{n}{\kappa_d}K_1(k_0\kappa_d r)\hat{\rho} + K_0(k_0\kappa_d r)\hat{z}\right]e^{ink_0 z}$$

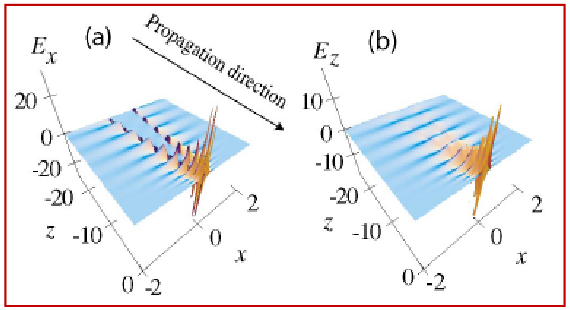
$$E_2(r < R, z) = A(z)\left[i\frac{n}{\kappa_m}I_1(k_0\kappa_m r)\hat{\rho} + I_0(k_0\kappa_m r)\hat{z}\right]e^{ink_0 z}$$



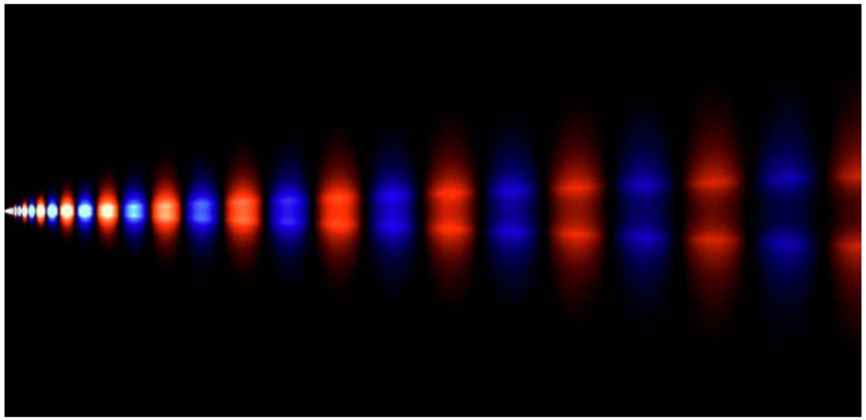
Plot the equations!

E_ρ

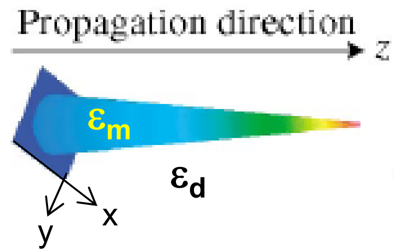




E_z



Dispersion relation of metal nanotips



$$\frac{\varepsilon_m}{\kappa_m}\frac{I_1(k_0\kappa_m R)}{I_0(k_0\kappa_m R)} + \frac{\varepsilon_d}{\kappa_d}\frac{K_1(k_0\kappa_d R)}{K_0(k_0\kappa_d R)} = 0 \qquad \kappa_m = \sqrt{n^2 - \varepsilon_m} \quad \kappa_d = \sqrt{n^2 - \varepsilon_d}. \qquad k_\parallel = n(z)k_0$$

For a thin, nanoscale-radius wire → $|k_\parallel R| \ll 1$ (corresponding to large $|\epsilon_2/\epsilon_1|$),

$$\frac{\epsilon_2}{\epsilon_1} = \frac{2}{(\gamma - \log 2 + \log C)(C)^2}, \longrightarrow \quad k_\parallel R = C, \quad k_\parallel = nk_0 \longrightarrow n(R) \approx \frac{1}{k_0 R}\sqrt{-\frac{2\varepsilon_d}{\varepsilon_m}}\left[\ln\sqrt{-\frac{4\varepsilon_m}{\varepsilon_d}} - \gamma\right]^{-1},$$

$\gamma \approx 0.577$ is Euler's constant.

For $k_0 R \to 0$, **the phase velocity** $v_p = c/n(z) \to 0$ **and the group velocity** $v_g = c/[d(n\omega)/d\omega] \to 0$

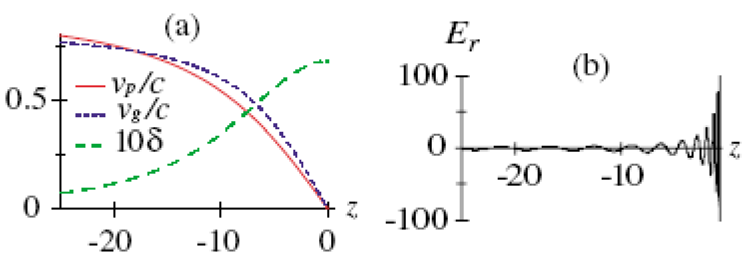
The time to reach the point R = 0 (or z = 0) $\propto \int n(R)dR \propto -\ln(k_0 R) \to \infty$

The eikonal parameter (also called WKB or adiabatic parameter) is defined as

$$\delta = |R' d(k_0 n)^{-1}/dR|, \text{ where } R' = dR/dz$$

For the applicability of the eikonal (WKB) approximation,
it necessary and sufficient that

$$\delta \ll 1$$



At the nanoscale tip of the wire, $\delta \approx |R'\sqrt{-\frac{\varepsilon_m}{2\varepsilon_d}}[\ln\sqrt{-\frac{4\varepsilon_m}{\varepsilon_d}} - \gamma]|$