13. Fresnel diffraction

Augustin Jean Fresnel (1788-1827)
Made contributions to transverse nature of light and diffraction theory

Josef von Fraunhofer (1787-1826)
Developed diffraction gratings and increased understanding of diffraction
Remind! Diffraction regimes

Full Wave Equations & Fresnel-Kirchoff

Rayleigh-Sommerfeld (near field)

Fresnel (far field)

Fraunhofer (far field)

Vector Solutions

(z, η)

Micro systems

850 nm 1550 nm

966 μm 791 μm

4.6 mm 2.5 mm

Examples: 50 μm Aperture, 200 μm Observation, λ=850 nm, λ=1550 nm

Fraunhofer Approximation - Assume planar wavefronts

Fresnel Approximation - Assume parabolic wavefronts

Rayleigh-Sommerfeld Formulation - Spherical wavefronts
Fresnel-Kirchhoff diffraction formula

\[ E(P_0) = \frac{E_0}{i\lambda} \iiint_{\Sigma} \frac{\exp(ikr)}{r} F(\theta) dA \]

**Obliquity factor:** \( F(\theta) = \cos \theta = \frac{z}{r} \)

\[ E(x, y) = \frac{zE_0}{i\lambda} \iiint_{\Sigma} \frac{\exp(ikr)}{r^2} d\xi d\eta \]

\[ r = \sqrt{z^2 + (x - \xi)^2 + (y - \eta)^2} \]

\[ \approx z \left[ 1 + \frac{1}{2} \left( \frac{x - \xi}{z} \right)^2 + \frac{1}{2} \left( \frac{y - \eta}{z} \right)^2 \right] = z + \frac{(x - \xi)^2}{2z} + \frac{(y - \eta)^2}{2z} \]

\[ = z + \left( \frac{x^2}{2z} + \frac{y^2}{2z} \right) + \left( \frac{\xi^2}{2z} + \frac{\eta^2}{2z} \right) - \left( \frac{x\xi}{z} + \frac{y\eta}{z} \right) \]

\[ E(x, y) = \frac{E_0}{i\lambda z} \exp(ikz) \exp \left[ i \frac{k}{2z} \left( x^2 + y^2 \right) \right] \]

\[ \times \iiint_{\Sigma} \exp \left[ i \frac{k}{2z} \left( \xi^2 + \eta^2 \right) \right] \exp \left[ -i \frac{k}{z} (x\xi + y\eta) \right] d\xi d\eta \]
\[ E(x, y) = \frac{E_0}{i\lambda z} \exp(ikz) \exp \left[ i \frac{k}{2z} (x^2 + y^2) \right] \]
\[ \times \iint_\Sigma \exp \left[ i \frac{k}{2z} (\xi^2 + \eta^2) \right] \exp \left[ -i \frac{k}{z} (x\xi + y\eta) \right] d\xi d\eta \]
\[ = C \iint_\Sigma \exp \left[ i \frac{k}{2z} (\xi^2 + \eta^2) \right] \exp \left[ -i \frac{k}{z} (x\xi + y\eta) \right] d\xi d\eta \]

**Fresnel diffraction**

\[ E(x, y) = C \iint U(\xi, \eta) \exp \left[ i \frac{k}{2z} (\xi^2 + \eta^2) \right] \exp \left[ -i \frac{k}{z} (x\xi + y\eta) \right] d\xi d\eta \]
\[ \Rightarrow E(x, y) \propto \mathcal{F} \left\{ U(\xi, \eta) e^{\frac{jk}{2z} (\xi^2 + \eta^2)} \right\} \]

**Fraunhofer diffraction**

\[ E(x, y) = C \iint U(\xi, \eta) \exp \left[ -i \frac{k}{z} (x\xi + y\eta) \right] d\xi d\eta \]
\[ = C \iint U(\xi, \eta) \exp \left[ -ik \left( \xi \sin \theta_\xi + \eta \sin \theta_\eta \right) \right] d\xi d\eta \]
\[ \Rightarrow E(x, y) \propto \mathcal{F} \left\{ U(\xi, \eta) \right\} \]
This is most general form of diffraction
- No restrictions on optical layout
  - near-field diffraction
  - curved wavefront
- Analysis somewhat difficult

\[
U(x, y) \approx \mathcal{F}\left\{ U(\xi, \eta)e^{\frac{j}{2z}(\xi^2 + \eta^2)} \right\}
\]
Accuracy of the Fresnel Approximation

\[
z^3 \gg \frac{\pi}{4\lambda} \left[ (x - \xi)^2 + (y - \eta)^2 \right]_{\text{max}}^2
\]

- Accuracy can be expected for much shorter distances

\[
\text{for } U(\xi,\eta) \text{ smooth & slow varying function; } 2|x - \xi| = D \leq 4\sqrt{\lambda z}
\]

\[
z \geq \frac{D^2}{16\lambda} \quad \text{Fresnel approximation}
\]
In summary, Fresnel diffraction is ...

Assume: \( z >> x_1, y_1; x_0, y_0 \)

\[
\int \int \rightarrow \int \int \quad (U = 0 \text{ outside the aperture})
\]

Fresnel's approximation:

In the exponent:

\[
r = \sqrt{z^2 + (x_0 - x_1)^2 + (y_0 - y_1)^2}
\]

\[
\approx z \left[ 1 + \frac{1}{2} \left( \frac{x_0 - x_1}{z} \right)^2 + \frac{1}{2} \left( \frac{y_0 - y_1}{z} \right)^2 \right] \quad \left( \sqrt{1+x} \approx 1 + \frac{1}{2}x \right)
\]

In the denominator: \( r \rightarrow z \)

\[
U(x_0, y_0) = -\frac{ik}{2\pi z} \int \int \int U(x_1, y_1) e^{\frac{ik}{2z} \left[ (z_0 - x_1)^2 + (y_0 - y_1)^2 \right]} \, dx_1 \, dy_1
\]
13-7. Fresnel Diffraction by Square Aperture

Fresnel Diffraction from a slit of width $D = 2a$. (a) Shaded area is the geometrical shadow of the aperture. The dashed line is the width of the Fraunhofer diffracted beam.

(b) Diffraction pattern at four axial positions marked by the arrows in (a) and corresponding to the Fresnel numbers $N_F=10$, 1, 0.5, and 0.1. The shaded area represents the geometrical shadow of the slit. The dashed lines at $|x| = (\lambda/D)d'$ represent the width of the Fraunhofer pattern in the far field. Where the dashed lines coincide with the edges of the geometrical shadow, the Fresnel number $N_F=0.5$.

$$N_F = \frac{a^2}{\lambda z} : \text{Fresnel number}$$
\[ U(x, y) = \frac{e^{ikz}}{j\lambda z} \int_{-\infty}^{\infty} \int_{\xi} U(\xi, \eta) \exp \left\{ j \frac{k}{2z} \left[ (x - \xi)^2 + (y - \eta)^2 \right] \right\} d\xi d\eta \]

The Fresnel Formula

\[ U_P = \frac{-ikU_0}{2\pi zz'} e^{ik|PS|} \int_{A} \exp \left( \frac{ik}{2z} \left[ (x_0 - x_m)^2 + (y_0 - y_m)^2 \right] \right) d\xi \]

- let \( z' \to \infty \) and use a point on the axis \( x, y = 0 \)

\[ U_P = B \int_{A} \exp \left( \frac{ik}{2z} \left[ x_0^2 + y_0^2 \right] \right) dS \]

\[ = B \int_{x_1}^{x_2} \exp \left( \frac{ikx_0^2}{2z} \right) dx_0 \int_{y_1}^{y_2} \exp \left( \frac{iky_0^2}{2z} \right) dy_0 \]

\[ = B \int_{x_1}^{x_2} \exp \left( \frac{i\pi u^2}{2} \right) dx_0 \int_{y_1}^{y_2} \exp \left( \frac{i\pi v^2}{2} \right) dy_0 \]

- where \( u^2 = \frac{Kx_0^2}{\pi z} \), \( v^2 = \frac{ky_0^2}{\pi z} \)
The Fresnel Integral

\[ \int_{s_1}^{s_2} \exp(i\pi w^2/2) dw = \int_{s_1}^{s_2} \cos(\pi w^2/2) dw + i \int_{s_1}^{s_2} \sin(\pi w^2/2) dw \]

\[ = C(s) + iS(s) \]

- Cornu spiral
- As \( s \to \pm \infty \), \( C(s), S(s) \to \pm \frac{1}{2} \)
- Value of integral can be evaluated numerically
Fresnel Integral Definitions

\[ I(\xi) \equiv \int_{0}^{\xi} e^{\frac{jt}{2} i t^2} dt \quad \text{[Complex Fresnel Integral \( I \)}} \]

\[ I(\xi) = C(\xi) + jS(\xi) \quad \text{[Fresnel Integrals \( C \) and \( S \)]} \]

\[ C(\xi) = \int_{0}^{\xi} \cos\left(\frac{\pi}{2} t^2\right) dt \]

\[ S(\xi) = \int_{0}^{\xi} \sin\left(\frac{\pi}{2} t^2\right) dt \]
Plot of Fresnel Integrals
\( \Delta v = 0.283 \)

\( \Delta v = 2.83 \)

\( \Delta v = 5.65 \)

\( \Delta v = 8.94 \)

\( \Delta v = N_F = \frac{a^2}{\lambda z} : \) Fresnel number
Fresnel diffraction from a wire
Fresnel diffraction from a straight edge
From Huygens’ principle to Fresnel-Kirchhoff diffraction
Huygens’ principle

Every point on a wave front is a source of secondary wavelets. i.e. particles in a medium excited by electric field (E) re-radiate in all directions i.e. in vacuum, E, B fields associated with wave act as sources of additional fields

Given wave-front at $t$

Allow wavelets to evolve for time $\Delta t$

New wavefront

Construct the wave front tangent to the wavelets

Secondary wavelet

$r = c \Delta t \approx \lambda$

What about $-r$ direction?
(π-phase delay when the secondary wavelets, Hecht, 3.5.2, 3rd Ed)
Huygens’ wave front construction

the electric field at \( P \) due to a superposition of all the Huygens’ wavelets from the wavefront at the aperture,

\[
dE_P = \left( \frac{dE_0}{r} \right) e^{ikr}
\]

\[
dE_0 \propto E_L \, da
\]

The amplitude \( E_L \) at point \( O \) is the amplitude of the spherical wave originating at the source,

\[
E_L = \left( \frac{E_s}{r'} \right) e^{ikr'}
\]

\[
\Rightarrow dE_P = \left( \frac{E_s}{rr'} \right) e^{ik(r+r')} \, da
\]

The field at \( P \) due to the secondary wavelets from the entire aperture is the surface integral

\[
E_P = E_s \int \int_{A_P} \left( \frac{1}{rr'} \right) e^{ik(r+r')} \, da
\]
Incompleteness of Huygens’ principle

\[ E_P = E_s \int \int \left( \frac{1}{rr'} \right) e^{ik(r+r')} \, da \]

incomplete in two ways

First, it does not take into account the \( F(\theta) \), obliquity factor, which attenuates the diffracted waves according to their direction.

Second, it does not take into account a curious requirement, a 90° phase shift of the diffracted waves relative to the primary incident wave.

Fresnel’s modification ➔ Huygens-Fresnel principle
Huygens-Fresnel principle

The Huygens-Fresnel principle. Each point on a wavefront generates a spherical wave.

\[ E_p = E_s \int \int \left( \frac{1}{rr'} \right) e^{ik(r-r')} \, da \]

Huygens’ Secondary wavelets on the wavefront surface O

Spherical wave from the point source S

Obliquity factor:
- unity where \( \theta = 0 \)
- zero where \( \theta = \pi/2 \)
Kirchhoff modification

Fresnel’s shortcomings:

He did not mention the existence of backward secondary wavelets, however, there also would be a reverse wave traveling back toward the source. He introduce a quantity of the obliquity factor, but he did little more than conjecture about this kind.

\[
E_p = E_s \frac{1}{r'} e^{ikr'} \int \int_{A_p} \frac{1}{r} e^{ikr} F(\theta) da, \quad \left( -\frac{\pi}{2} < \theta < \frac{\pi}{2} \right)
\]

Gustav Kirchhoff: Fresnel-Kirchhoff diffraction theory

A more rigorous theory based directly on the solution of the differential wave equation. He, although a contemporary of Maxwell, employed the older elastic-solid theory of light. He found \( F(\theta) = (1 + \cos \theta)/2 \).

\( F(0) = 1 \) in the forward direction, \( F(\pi) = 0 \) with the back wave.

**Fresnel-Kirchhoff diffraction formula**

\[
E_p = \frac{-ikE_s}{2\pi} \int \int F(\theta) e^{ik(r+r')} \frac{1}{rr'} da
\]

where the factor \( -i = e^{-i\pi/2} \) represents the required phase shift, and \( F(\theta) = \frac{1 + \cos \theta}{2} \).
Fresnel-Kirchhoff diffraction integral

\[ E_p = \frac{-ikE_s}{2\pi} \iint_{A_p} \left\{ \frac{1 + \cos \theta}{2} \right\} \frac{1}{rr'} e^{ik(r+r')} \, da, \quad ( -\pi < \theta < \pi ) \]

Arnold Johannes Wilhelm Sommerfeld: Rayleigh-Sommerfeld diffraction theory
A very rigorous solution of partial differential wave equation.
The first solution utilizing the electromagnetic theory of light.

This final formula looks similar to the Fresnel-Kirchhoff formula, therefore, now we call this the revised Fresnel-Kirchhoff formula, or, just call the Fresnel-Kirchhoff diffraction integral.
HUYGENS-FRESNEL CONSTRUCTION: Fresnel Zones

The total contribution to the disturbance at $P$ is expressed as an area integral over the primary wavefront.

$$\psi(P) = A \frac{\exp\left[-i(\omega t - kr_0)\right]}{r_0} \int_S \int_S \exp(iks) K(\chi) dS$$

(8)

Spherical wave from source $P_0$

Huygens' Secondary wavelets on the wavefront surface $S$

Obliquity factor:
- unity where $\chi=0$ at $C$
- zero where $\chi=\pi/2$ at high enough zone index

**Figure 2** Fresnel zone construction. $P_0$: point source, $S$: wavefront, $r_0$: radius of the wavefront, $b$: distance $CP$, $s$: distance $QP$. (After Born and Wolf.)
\[
\psi(P) = A \frac{\exp \left[ -i(\omega t - kr_0) \right]}{r_0} \int_S \frac{\exp \left( ik \chi \right)}{s} K(\chi) \, dS
\]

The average distance of successive zones from \( P \) differs by \( \lambda/2 \rightarrow \) half-period zones. Thus, the contributions of the zones to the disturbance at \( P \) alternate in sign,

\[
\psi(P) = \psi_1 - \psi_2 + \psi_3 - \psi_4 + \psi_5 - \psi_6 + \ldots
\]

where \( \psi_j \) stands for the contribution of the \( j \)th zone, \( j = 1, 2, 3, \ldots \). The contribution of each annular zone is directly proportional to the zone area and is inversely proportional to the average distance of the zone to the point of observation \( P \). The ratio of the zone area to its average distance from \( P \) is independent of the zone index \( j \). Thus, in summing the contributions of the zones we are left with only the variation of the obliquity factor, \( K(\chi) \). To a good approximation, the obliquity factors for any two adjacent zones are nearly equal and for a large enough zone index \( j \) the obliquity factor becomes negligible. The total disturbance at the point of observation \( P \) may be approximated by

\[
\psi(P) = 1/2(\psi_1 \pm \psi_n)
\]

(1/2 means averaging of the possible values, more details are in 10-3, Optics, Hecht, 2nd Ed)

For an unobstructed wave, the last term \( \psi_n = 0 \).

\[
\psi(P) = 1/2\psi_1
\]

\[
= \frac{A}{r_0 + b} \lambda \exp \left\{ -i[\omega t - k(r_0 - b) - \pi/2] \right\}
\]

Whereas, a freely propagating spherical wave from the source \( P_0 \) to \( P \) is

\[
\psi(P) = \frac{A}{r_0 + b} \exp \left\{ -i[\omega t - k(r_0 + b)] \right\}
\]

Therefore, one can assume that the complex amplitude of \( \exp(iks)/s \)

\[
= \frac{1}{i\lambda} \left( \frac{\exp(iks)}{s} \right)
\]
(a) The first two zones are uncovered,
\[ \psi(P) = \psi_1 - \psi_2 = 0! \]  (consider the point P at the on-axis P)
since these two contributions are nearly equal.

(b) The first zone is uncovered if point P is placed farther away,
\[ \psi(P) = \psi_1 \]
which is twice what it was for the unobstructed wave!

(c) Only the first zone is covered by an opaque disk,
\[ \psi(P) = -\psi_2 + \psi_3 - \psi_4 + \psi_5 - \psi_6 + \cdots = -\frac{1}{2}\psi_2 \approx \frac{1}{2}\psi_1 \]
which is the same as the amplitude of the unobstructed wave.
Fresnel diffraction from a circular aperture
**Babinet principle**

\[ \psi_S(P) + \psi_{CS}(P) = \psi_{UN}(P) \]
HUYGENS-FRESNEL CONSTRUCTION : Straight edge

\( \psi_a(P) = \frac{1}{2} \psi_4 \)

At the edge

\( \psi_b(P) = \frac{1}{2} \psi_1 + \psi_1 = \frac{3}{2} \psi_1 \)

\( \psi(P) = \frac{3}{2} \psi_1 - \psi_2 \)

Monotonically decreasing

\( \psi_c(P) = -\frac{1}{2} \psi_2 \)

\( \psi(P) = \frac{1}{2} \psi_3 \)

Damped oscillating
13-6. The Fresnel zone plate

The average distance of successive zones from P differs by $\lambda/2 \rightarrow$ half-period zones. Thus, the contributions of the zones to the disturbance at P alternate in sign,

$$\psi(P) = \psi_1 - \psi_2 + \psi_3 - \psi_4 + \psi_5 - \psi_6 + \ldots$$

Assume plane wavefronts

$$R_n^2 = \left( r_0 + \frac{n\lambda}{2} \right)^2 - r_0^2 = r_0^2 \left[ n \frac{\lambda}{r_0} + \frac{n^2}{4} \left( \frac{\lambda}{r_0} \right)^2 \right]$$

$$R_n \approx \sqrt{nr_0\lambda} \quad (r_0 >> \lambda)$$

If the even zones $(n=even)$ are blocked

$$\psi(P) = \psi_1 + \psi_3 + \psi_5 + \ldots$$

Bright spot at P

It acts as a lens!
Fresnel zone-plate lens

\[ R_n \approx \sqrt{nr_0 \lambda} \quad (r_0 >> \lambda) \]

\[ r_0 = \frac{R_n^2}{n\lambda} \]

\[ f_1 = r_0 (n = 1) = \frac{R_1^2}{\lambda} \]

Fresnel zone-plate lens has multiple foci.

\[ \Delta R_n = \sqrt{r_0 \lambda} \frac{\Delta n}{2\sqrt{n}} = R_1 \frac{1}{2\sqrt{n}} \]

\[ (\Delta R_n) \sin \theta_m = m\lambda \Rightarrow \sin \theta_m \sim \tan \theta_m = \frac{R_n}{f_m} = \frac{m\lambda}{\Delta R_n} \]

\[ f_m = (R_n)(\Delta R_n) \frac{1}{m\lambda} = (\sqrt{nR_1})\left(\frac{R_1}{2\sqrt{n}}\right) \frac{1}{m\lambda} \]

\[ f_m = \frac{R_1^2}{m\lambda} \]
**Fresnel zone-plate lens**

Binary zone plate:
The areas of each ring, both light and dark, are equal. It has multiple focal points.

Sinusoidal zone plate:
This type has a single focal point.

Fresnel lens:
This type has a single focal point. Focusing efficiency approaches 100%.