19. Theory of Multilayer Films

Transfer Matrix

Reflectance at Normal Incidence

Anti-reflecting Films

High-Reflectance Films
Consider two waves $E$ and $E'$ that have the same frequency $\omega$.

$$E' = E - E = \omega \varepsilon$$

$$E' = E - E = \omega \varepsilon$$

$$E' = E - E = \omega \varepsilon$$

$$E' = E - E = \omega \varepsilon$$

Figure 19-1  Reflection of a beam from a single layer. The diagram defines quantities used in applying boundary conditions to write Eqs. (19-6)-(19-9). Note that a bold dot is used to denote directions perpendicular to the plane of incidence.
\[ E_{r1} = \sum_{m=1}^{\infty} E_{r1}^m \]

\[ E_{i1} = \sum_{m=1}^{\infty} E_{i1}^m \]

\[ E_{i2} = \sum_{m=1}^{\infty} E_{i2}^m \]

\[ E_{r2} = \sum_{m=1}^{\infty} E_{r2}^m \]

\[ \vec{E}_0 = E_z \hat{z} \text{ (TE - mode)} \]

\[ \vec{B}_0 = B_x \hat{x} + B_y \hat{y} \text{ (TE - mode)} \]
Boundary conditions for the electric and magnetic fields of plane waves incident on the interfaces (a) and (b) are simply stated:

Tangential components of $E$ and $B$-fields are continuous across the interface.

On (a): \[ E_a = E_0 + E_{r1} = E_{t1} + E_{i1} \]
\[ B_a = B_0 \cos \theta_0 - B_{r1} \cos \theta_0 = B_{t1} \cos \theta_{t1} - B_{i1} \cos \theta_{t1} \]

On (b): \[ E_b = E_{i2} + E_{r2} = E_{t2} \]
\[ B_b = B_{i2} \cos \theta_{t1} - B_{r2} \cos \theta_{t1} = B_{t2} \cos \theta_{t2} \]
Transfer Matrix for a Single Film on a Substrate: TE Mode

Note:

\[ B = \frac{E}{v} = \left(\frac{n}{c}\right) E = n\sqrt{\varepsilon_0 \mu_0} E \]

Define:

\[ \gamma_0 = n_0 \sqrt{\varepsilon_0 \mu_0} \cos \theta_0 \]
\[ \gamma_1 = n_1 \sqrt{\varepsilon_0 \mu_0} \cos \theta_{t1} \]
\[ \gamma_s = n_2 \sqrt{\varepsilon_0 \mu_0} \cos \theta_{t2} \]

(a) \[ B_a = B_0 \cos \theta_0 - B_{r1} \cos \theta_0 = B_{t1} \cos \theta_{t1} - B_{i1} \cos \theta_{t1} \]
(b) \[ B_b = B_{t2} \cos \theta_{t1} - B_{r2} \cos \theta_{t1} = B_{t2} \cos \theta_{t2} \]

\[ B_a = \gamma_0 \left( E_0 - E_{r1} \right) = \gamma_1 \left( E_{t1} - E_{i1} \right) \]
\[ B_b = \gamma_1 \left( E_{i2} - E_{r2} \right) = \gamma_s \ E_{t2} \]
From the geometry of the problem we can see that:

\[ E_{i2} = E_{i1} \exp(-i\delta) \quad \text{and} \quad E_{i1} = E_{r2} \exp(-i\delta) \]

where \( \delta = k_0 \Delta = \left( \frac{2\pi}{\lambda_0} \right) n_1 t \cos \theta_{i1} \).

Using these expressions we find:

\[ E_b = E_{i2} + E_{r2} = E_{i1} \exp(-i\delta) + E_{i1} \exp(+i\delta) = E_{t2} \]

\[ B_b = \gamma_1 (E_{i2} - E_{r2}) = \gamma_1 \left[ E_{i1} \exp(-i\delta) - E_{i1} \exp(+i\delta) \right] = \gamma_s E_{t2} \]
Transfer Matrix for a Single Film on a Substrate: TE Mode

We can solve the last two equations for $E_{i1}$ and $E_{i1}$ in terms of $E_b$ and $B_b$:

$$E_{i1} = \left( \frac{\gamma_1 E_b + B_b}{2 \gamma_1} \right) \exp(i \delta)$$

$$E_{i1} = \left( \frac{\gamma_1 E_b - B_b}{2 \gamma_1} \right) \exp(-i \delta)$$

At the first boundary we then obtain:

$$E_a = E_{r1} + E_{i1} = E_b \left[ \frac{\exp(+i \delta) + \exp(-i \delta)}{2} \right] + B_b \left[ \frac{\exp(+i \delta) - \exp(-i \delta)}{2 \gamma_1} \right]$$

$$= E_b \cos \delta + B_b \left( \frac{i \sin \delta}{\gamma_1} \right)$$

$$B_a = \gamma_1 (E_{r1} - E_{i1}) = \gamma_1 E_b \left[ \frac{\exp(+i \delta) - \exp(-i \delta)}{2} \right] + \gamma_1 B_b \left[ \frac{\exp(+i \delta) + \exp(-i \delta)}{2 \gamma_1} \right]$$

$$= E_b (\gamma_1 i \sin \delta) + B_b \cos \delta$$
Transfer Matrix for a Single Film on a Substrate: TE Mode

\[ E_a = E_b \cos \delta + B_b \left( \frac{i \sin \delta}{\gamma_1} \right) \]
\[ B_a = E_b \left( \gamma_1 i \sin \delta \right) + B_b \cos \delta \]

This can be written in matrix form:

\[
\begin{bmatrix}
E_a \\
B_a
\end{bmatrix} =
\begin{bmatrix}
\cos \delta & i \sin \delta \\
\gamma_1 i \sin \delta & \cos \delta
\end{bmatrix}
\begin{bmatrix}
E_b \\
B_b
\end{bmatrix}
\]

The 2x2 transfer matrix is given by:

\[ \mathcal{M} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \]

Generalizing to a system with \( N \) layers:

\[
\begin{bmatrix}
E_a \\
B_a
\end{bmatrix} = \mathcal{M}_1 \mathcal{M}_2 \mathcal{M}_3 \ldots \mathcal{M}_N
\begin{bmatrix}
E_N \\
B_N
\end{bmatrix}
= \mathcal{M}_N
\begin{bmatrix}
E_N \\
B_N
\end{bmatrix}
\]

for a multilayer
In terms of \( E_0, E_{r1}, \) and \( E_{t2} \):

\[
E_a = E_0 + E_{r1} \quad E_b = E_{t2} \\
B_a = \gamma_0 (E_0 - E_{r1}) \quad B_b = \gamma_s E_{t2}
\]

In matrix form we obtain:

\[
\begin{bmatrix}
E_0 + E_{r1} \\
\gamma_0 (E_0 - E_{r1})
\end{bmatrix} =
\begin{bmatrix}
\cos \delta & i \sin \delta \\
\gamma_1 i \sin \delta & \cos \delta
\end{bmatrix}
\begin{bmatrix}
E_{t2} \\
\gamma_s E_{t2}
\end{bmatrix} =
\begin{bmatrix}
m_{11} & m_{12} \\
m_{21} & m_{22}
\end{bmatrix}
\begin{bmatrix}
E_{t2} \\
\gamma_s E_{t2}
\end{bmatrix}
\]

The amplitude reflection and transmission coefficients are given by:

\[
r \equiv \frac{E_{r1}}{E_0} \quad t \equiv \frac{E_{t2}}{E_0}
\]

In terms of \( r \) and \( t \) we obtain:

\[
1 + r = m_{11} t + m_{12} \gamma_s t \\
\gamma_0 (1 - r) = m_{12} t + m_{22} \gamma_s t
\]
Solving for $r$ and $t$ we have:

$$t = \frac{2\gamma_0}{\gamma_0 m_{11} + \gamma_0 \gamma_s m_{12} + m_{21} + \gamma_s m_{22}}$$

$$r = \frac{\gamma_0 m_{11} + \gamma_0 \gamma_s m_{12} - m_{21} - \gamma_s m_{22}}{\gamma_0 m_{11} + \gamma_0 \gamma_s m_{12} + m_{21} + \gamma_s m_{22}}$$

These equations allow us to evaluate the properties of multilayer films.
19-2. Reflectance at Normal Incidence: Single-Layer Film

At normal incidence the matrix elements become:

\[ m_{11} = \cos \delta \]
\[ m_{12} = \frac{i \sin \delta}{n_1 \sqrt{\varepsilon_0 \mu_0}} \]
\[ m_{21} = i n_1 \sqrt{\varepsilon_0 \mu_0} \sin \delta \]
\[ m_{22} = \cos \delta \]

At normal incidence the reflectance coefficient becomes:

\[ r = \frac{n_1 (n_0 - n_s) \cos \delta + i (n_0 n_s - n_1^2) \sin \delta}{n_1 (n_0 + n_s) \cos \delta + i (n_0 n_s + n_1^2) \sin \delta} \]

\[ \delta = \frac{2\pi}{\lambda_0} n_1 t \cos \theta_{t1} = \frac{2\pi}{\lambda_0} n_1 t \]

The power reflectance coefficient is given by:

\[ R = r^* r = \frac{n_1^2 (n_0 - n_s)^2 \cos^2 \delta + (n_0 n_s - n_1^2)^2 \sin^2 \delta}{n_1^2 (n_0 + n_s)^2 \cos^2 \delta + (n_0 n_s + n_1^2)^2 \sin^2 \delta} \]

(Reflectance)
$\Delta = \frac{\lambda}{4} = \frac{\lambda_0}{4n}$

$n_0 = 1$ (air)

$n_1 = n$

$n_2 = n_s = 1.52$ (glass)

$\Delta / \lambda = nt / \lambda_0$

Fig. 19-2 Reflectance from a single film layer versus normalized path difference. The dashed line represents the uncoated glass substrate of index $n_s = 1.52$. 
Reflectance at Normal Incidence: Single-Layer Quarter-Wave Film

For the quarter-wave thickness $t = \frac{\lambda}{4} = \frac{\lambda_0}{4n_1}$:

$$\delta = \frac{2\pi n_1 t}{\lambda_0} = \frac{2\pi n_1}{\lambda_0} \left( \frac{\lambda_0}{4n_1} \right) = \frac{\pi}{2} \rightarrow \cos \delta = 0, \sin \delta = 1$$

$$m_{11} = \cos \delta = 0 \quad m_{12} = \frac{i \sin \delta}{n_1 \sqrt{\varepsilon_0 \mu_0}} = \frac{i}{n_1 \sqrt{\varepsilon_0 \mu_0}} \Rightarrow \begin{pmatrix} 0 & i \\ i \gamma_1 & 0 \end{pmatrix}$$

$$m_{21} = i n_1 \sqrt{\varepsilon_0 \mu_0} \sin \delta = i n_1 \sqrt{\varepsilon_0 \mu_0} \quad m_{22} = \cos \delta = 0$$

At normal incidence the reflectance coefficient becomes:

$$r = \frac{i (n_0 n_s - n_i^2)}{i (n_0 n_s + n_i^2)} = \frac{\left( \frac{n_0 n_s - n_i^2}{n_0 n_s + n_i^2} \right)}{\left( \frac{n_0 n_s + n_i^2}{n_0 n_s + n_i^2} \right)}$$

The power reflectance coefficient is given by:

$$R = r r^* = \frac{\left( n_0 n_s - n_i^2 \right)^2}{\left( n_0 n_s + n_i^2 \right)^2}$$

$$t = \frac{\lambda}{4} = \frac{\lambda_0}{4n_1} \quad n_1 = \sqrt{n_0 n_s} \quad R = 0$$

A single $\lambda/4$ film can produce perfect antireflection!
Reflectance at Normal Incidence: Two-Layer Quarter-Wave Films

The transfer matrix for a quarter-wave thickness film is given by: \( \mathbf{M} = \begin{bmatrix} 0 & i \\ i \gamma_1 & 0 \end{bmatrix} \)

For two quarter-wave films:

\[
\mathbf{M} = \mathbf{M}_1 \mathbf{M}_2 = \begin{bmatrix} 0 & i \\ i \gamma_1 & 0 \end{bmatrix} \begin{bmatrix} 0 & i \\ i \gamma_1 & 0 \end{bmatrix} = \begin{bmatrix} -\gamma_2 \gamma_1 & 0 \\ 0 & -\gamma_1 \gamma_2 \end{bmatrix}
\]

The reflectance coefficient becomes:

\[
r = \frac{\gamma_0 m_{11} + \gamma_0 \gamma_s m_{12} - \gamma_s m_{21} - \gamma_s m_{22}}{\gamma_0 m_{11} + \gamma_0 \gamma_s m_{12} + \gamma_s m_{21} + \gamma_s m_{22}} = \frac{\gamma_0 \left( -\frac{\gamma_2}{\gamma_1} \right) - \gamma_s \left( -\frac{\gamma_1}{\gamma_2} \right)}{\gamma_0 \left( -\frac{\gamma_2}{\gamma_1} \right) + \gamma_s \left( -\frac{\gamma_1}{\gamma_2} \right)} = \frac{\gamma_2 \gamma_0 - \gamma_s \gamma_1^2}{\gamma_2 \gamma_0 + \gamma_s \gamma_1^2} = \frac{n_2^2 n_0 - n_s n_1^2}{n_2^2 n_0 + n_s n_1^2}
\]

The power reflectance coefficient is given by:

\[
R = rr^\ast = \frac{\left( n_2^2 n_0 - n_s n_1^2 \right)}{\left( n_2^2 n_0 + n_s n_1^2 \right)^2}
\]

Zero reflectance occurs when \( n_2^2 n_0 - n_s n_1^2 = \frac{n_2}{n_1} = \sqrt{\frac{n_s}{n_0}} \)
Reflectance at Normal Incidence: Two-Layer Quarter-Wave Films

Although the reflectance is greater, it remains less values over the broad range by introducing a $\lambda/2$ layer!
Reflectance at Normal Incidence: Two-Layer Films

Figure 19-5  Antireflecting double layer using $\lambda/4 - \lambda/2$ thickness films. Reflectance curves are shown in Figure 19-4.

<table>
<thead>
<tr>
<th>TABLE 19-1</th>
<th>REFRACTIVE INDICES FOR SEVERAL COATING MATERIALS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Material</td>
<td>Visible ($\sim 550$ nm)</td>
</tr>
<tr>
<td>Cryolite</td>
<td>1.30–1.33</td>
</tr>
<tr>
<td>MgF$_2$</td>
<td>1.38</td>
</tr>
<tr>
<td>SiO$_2$</td>
<td>1.46</td>
</tr>
<tr>
<td>SiO</td>
<td>1.55–2.0</td>
</tr>
<tr>
<td>Al$_2$O$_3$</td>
<td>1.60</td>
</tr>
<tr>
<td>CeF$_3$</td>
<td>1.65</td>
</tr>
<tr>
<td>ThO$_2$</td>
<td>1.8</td>
</tr>
<tr>
<td>Nd$_2$O$_3$</td>
<td>2.0</td>
</tr>
<tr>
<td>ZrO$_2$</td>
<td>2.1</td>
</tr>
<tr>
<td>CeO$_2$</td>
<td>2.35</td>
</tr>
<tr>
<td>ZnS</td>
<td>2.35</td>
</tr>
<tr>
<td>TiO$_2$</td>
<td>2.4</td>
</tr>
<tr>
<td>Si</td>
<td>—</td>
</tr>
<tr>
<td>Ge</td>
<td>—</td>
</tr>
</tbody>
</table>
19-4. Reflectance at Normal Incidence: Three-Layer Antireflecting Films

![Diagram of antireflecting triple layers](a) Quarter-quarter-quarter wavelength layers. (b) Quarter-half-quarter wavelength layers. Reflectance curves are shown in Figure 19-7.
Reflectance at Normal Incidence: Three-Layer Antireflecting Films

For the quarter-wave film, zero reflectance occurs when:

\[
\frac{n_1 n_3}{n_2} = \sqrt{n_0 n_s}
\]

Figure 19-7 Reflectance from triple-layer films versus wavelength. In all cases \(n_0 = 1\) and \(n_s = 1.52\). Thicknesses are determined at \(\lambda = 550\) nm. (a) \(\lambda/4-\lambda/4-\lambda/4\): \(n_1 = 1.38, n_2 = 2.02, n_3 = 1.8\). (b) \(\lambda/4-\lambda/2-\lambda/4\): \(n_1 = 1.38, n_2 = 2.2, n_3 = 1.7\).
CVI Antireflecting Films

1. Single Layer MgF₂ Coatings

- R < 0.25%, 0°, over wide ranges on sapphire, Nd:YAG, and high index glasses
- R < 0.25%, 45°P on glass and fused silica
CVI Broadband Antireflecting Films

193-248nm, 0°

248-355nm, 0°

425-675nm, 0°

670-1064nm, 0°
19-5. High-Reflectance Layers

A high-reflectance film rather than an anti-reflecting film is made by reversing the order of deposition of the low- and high refractive index layers.

The transfer matrix for a quarter-wave thickness film is given by:

\[ M = \begin{bmatrix} 0 & i \\ i \gamma & 0 \end{bmatrix} \]

For two quarter-wave films, with the high-refractive index layer on top:

\[ M_{HL} = M_H M_L = \begin{bmatrix} 0 & \frac{i}{\gamma_H} \\ i \gamma_H & 0 \end{bmatrix} \begin{bmatrix} 0 & i \\ i \gamma_L & 0 \end{bmatrix} = \begin{bmatrix} -\frac{\gamma_L}{\gamma_H} & 0 \\ 0 & -\frac{\gamma_H}{\gamma_L} \end{bmatrix} \]

The power reflectance coefficient for the two-layer film at normal incidence is given by:

\[ R = r^* r = \left( \frac{n_L^2 n_0 - n_s n_H^2}{n_L^2 n_0 + n_s n_H^2} \right)^2 \]
High-Reflectance stack of N Double-Layers

For a high-reflectance film with \( N \) pairs of high- and low refractive index layers:

\[
\mathcal{M} = (\mathcal{M}_H^1 \mathcal{M}_L^1)(\mathcal{M}_H^2 \mathcal{M}_L^2)\ldots(\mathcal{M}_H^N \mathcal{M}_L^N) = (\mathcal{M}_H \mathcal{M}_L)^N = \mathcal{M}_{HL}^N
\]

\[
\mathcal{M} = \begin{bmatrix} -\gamma_L & 0 \\ \gamma_H & 0 \end{bmatrix}^N = \begin{bmatrix} -\left(\frac{\gamma_L}{\gamma_H}\right)^N \\ \left(\frac{\gamma_H}{\gamma_L}\right)^N \end{bmatrix}
\]

The amplitude and power reflectance coefficients for the multi-layer film at normal incidence are given by:

\[
r = \frac{n_0 \left( -\frac{n_L}{n_H} \right)^N - n_s \left( -\frac{n_H}{n_L} \right)^N}{n_0 \left( -\frac{n_L}{n_H} \right)^N + n_s \left( -\frac{n_H}{n_L} \right)^N} = \left( \frac{n_0}{n_s} \right) \left( -\frac{n_L}{n_H} \right)^{2N} - 1
\]

\[
R = r r^* = \left[ \frac{(n_0/n_s)(n_L/n_H)^{2N} - 1}{(n_0/n_s)(n_L/n_H)^{2N} + 1} \right]^2
\]

Example

A high-reflectance stack like that of Figure 19-8 incorporates six double layers of SiO\(_2\) (\( n = 1.46 \)) and ZnS (\( n = 2.35 \)) films on a glass (\( n = 1.48 \)) substrate. What is the reflectance for light of 550 nm at normal incidence?

Solution

Substituting directly into Eq. (19-53), we get

\[
R = \left[ \frac{(1/1.48)(1.46/2.35)^{2N} - 1}{(1/1.48)(1.46/2.35)^{2N} + 1} \right]^2
\]

or \( R = 99.1\% \).
Figure 19-9 Spectral reflectance of a high-low index stack for (a) $N = 2$ and (b) $N = 6$ double layers. Curve (c) represents an $N = 2$ stack with an additional high-index layer adjacent to the substrate. Layers are $\lambda/4$ thick at $\lambda = 550$ nm. In all cases, $n_H = 2.35$, $n_L = 1.38$, $n_i = 1.52$, and $n_0 = 1.00$. 
### Reflectance of high-low $\lambda/4$ Layers

#### TABLE 19-2  REFLECTANCE OF A HIGH-LOW QUARTER-WAVE STACK

<table>
<thead>
<tr>
<th>$n_L/n_H$</th>
<th>$R$ (%)</th>
<th>$N$</th>
<th>$R$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>4.26</td>
<td>1</td>
<td>39.71</td>
</tr>
<tr>
<td>0.91</td>
<td>21.01</td>
<td>2</td>
<td>73.08</td>
</tr>
<tr>
<td>0.83</td>
<td>40.82</td>
<td>3</td>
<td>89.77</td>
</tr>
<tr>
<td>0.77</td>
<td>57.77</td>
<td>4</td>
<td>96.35</td>
</tr>
<tr>
<td>0.71</td>
<td>70.44</td>
<td>5</td>
<td>98.72</td>
</tr>
<tr>
<td>0.67</td>
<td>79.35</td>
<td>6</td>
<td>99.56</td>
</tr>
<tr>
<td>0.625</td>
<td>85.48</td>
<td>7</td>
<td>99.85</td>
</tr>
<tr>
<td>0.59</td>
<td>89.67</td>
<td>8</td>
<td>99.95</td>
</tr>
<tr>
<td>0.56</td>
<td>92.55</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.53</td>
<td>94.56</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.50</td>
<td>95.97</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
CVI High-Reflectance Coatings

Gas Laser Mirrors

Contact a CVI applications engineer for OEM mirror mounts and system integration capabilities.

λ/10 at 633nm over 85% of diameter coated surface figure available.

Reflectivity vs. Wavelength of AR4 Series 244-257nm Ion Laser Mirror at 0° and 45° incidence angles.
CVI High-Reflectance Coatings

Reflectivity vs. Wavelength of HC2 Series
325nm He:Cd Laser Mirror at 0° and 45° incidence angles.

Reflectivity vs. Wavelength of N Series
337nm Nitrogen Laser Mirror at 0° and 45° incidence angles.

Reflectivity vs. Wavelength of AR3 Series
351-364nm Ion Laser Mirror at 0° and 45° incidence angles.