GMR and Topological Photonics
Seok Ho Song (HYU)

- The first-order photonic stopband has widely been utilized for photonic crystal applications.
- At higher-order stopbands including second-order one, the coupling between GMR (leaky) band-edge states and a continuum of radiation states from the higher-order stopband can generate an confined mode, which can guide through the topological interface (edge).
- There may exist a strong relation between EP in NH photonics and DP in topological photonics.

“It is worth mentioning the potential of utilizing exceptional point singularities in optical scattering problems, where the coupling between discrete localized metastable states and a continuum of radiation states is concerned. Such optical scattering problems in general, be treated as non-Hermitian problems, for which a point of interest would be to explore the connection between radiation leakage and exceptional points emerging in the continuum.” (Mohammad-Ali Miri and Andrea Alù, Science 363, 42 (2019))
Simultaneous multi-frequency topological edge modes between one-dimensional photonic crystals

Optics Letters 41, 1644 (2016)

there are strongly localized fields near the interface between X and Y'.

The enhancement at the edge mode frequencies are about 20 to 30.

The enhancement can increase exponentially with increasing number of units. This result suggests that the interface region is suitable for inserting single or few layers of thin or 2-D materials for enhancing frequency upconversion optical processes.
Dispersion relation (photonic band structure)

Visualization of the vacuum band structure (2d)

For a two-dimensional system:

\[ \omega = c \sqrt{k_x^2 + k_y^2} \]

This function depicts a cone: light cone.

A few ways to visualize this band structure:

- Constant frequency contour
- Projected band diagram
- Band diagram along several “special” directions

Photonic Bandgap Formation

1. Dispersion curve for free space

2. In a periodic system, when half the wavelength corresponds to the periodicity

\[ \lambda/2 = a \quad k = \pi/a \]
the Bragg effect prohibits photon propagation.

3. At the band edges, standing waves form, with the energy being either in the high or the low index regions

4.Standing waves transport no energy with zero group velocity

Band-edges

Air band

standing wave in \( n_2 \)

Bandgap

standing wave in \( n_1 \)

Dielectric band

Reduced (1st Brillouin zone) band structure

\[ n_1 \text{ high index material}\]

\[ n_2 \text{ low index material}\]
Guided-mode resonance (GMR)
from symmetric and asymmetric grating profiles

(a) Grating with symmetric profile

(b) Grating with asymmetric profile

Details: Y. Ding and R. Magnusson, “Use of nondegenerate resonant leaky modes to fashion diverse optical spectra,” Optics Express, May 3, 2004
For symmetrical grating (Type I):

Non-leaky mode (out of phase, $\pi$)

leaky mode in phase

For asymmetrical grating (Type II):

leaky mode in phase

leaky mode in phase

Stop bands in Symmetric and Asymmetric profiles

Use of nondegenerate resonant leaky modes to fashion diverse optical spectra

Y. Ding and R. Magnusson

Opt Express 12, 1885 (2004)
Band crossing (Dirac cones),
Band flips,
Flatbands
in photonic lattices
It is possible to control the width of the 2nd-order band gap by lattice design. In particular, as a modal band closes, there results a degenerate state (band crossing) and the degenerate point executes a band flip.

For symmetrical grating (Type I):

Non-leaky modes via bound-states in the continuum (BIC)
leaky modes via guided-mode resonances (GMRs)
Band flips and bound-state transitions in leaky-mode photonic lattices
Sun-Goo Lee and Robert Magnusson

Semi-analytical calculation of dispersion relations

\[
\frac{\Delta \omega}{V_g} = \frac{1}{2} \left( \frac{g - \alpha}{2} \right) - i h_1 \pm [(h_2 + i h_1)^2 + k^2]^{-\frac{1}{2}}
\]

\[
\frac{\Delta \omega}{cK} = \left( -i h_1 \pm [(h_2 + i h_1)^2 + k^2]^{-\frac{1}{2}} \right) \frac{V_g}{cK} 
\]

\[
\Omega(k_z) = \Omega_0 - \left( i h_1 \pm \sqrt{k_z^2 + (h_2 + i h_1)^2} \right) / (K h_0),
\]

\[
\Omega = \Omega_{\text{Re}} + i \Omega_{\text{Im}}
\]

In order to calculate the three coupling coefficients \( h_0, h_1, \) and \( h_2, \)
the grating layer is treated as a homogeneous waveguide.

The mode profiles are given by

\[
\phi(x) = \begin{cases} 
N e^{-j\varphi} & (x > 0) \\
N(\text{Ae}^{j\varphi} + \text{Be}^{-j\varphi}) & (x \geq 0 > -d) \\
N e^{j\varphi} & (-d \geq x).
\end{cases}
\]

\[
N^2 = 2 p_2^2 \left[ \left( \frac{1}{\gamma_1 + 1/\gamma_1 + d} \right) - \left( p_1^2 + p_2^2 \right) \right] \text{ by normalization } \int_{-d}^{0} \phi(x) \phi^*(x) \, dx = 1.
\]

\[
A = (p_2 + i \gamma_1) / 2 p_2, \quad B = (p_2 - i \gamma_1) / 2 p_2, \quad C = (p_2 + i \gamma_1) / (p_2 - i \gamma_1) e^{j(\varphi_2 - \varphi_1)}.
\]

The Green's functions are given by

\[
G(x, x') = \begin{cases} 
t_{21}(e^{-j\varphi x} + r_{21} e^{j(\varphi x + \beta_1^2 x)} e^{-j\varphi x}) & (x > 0) \\
t_{21}(e^{-j\varphi x} + r_{21} e^{j\varphi x} e^{-j(\varphi x + \beta_1^2 x)}) & (0 \geq x > s) \\
t_{21}(e^{-j\varphi x} + r_{21} e^{j(\varphi x + \beta_1^2 x)} e^{-j\varphi x}) & (0 \geq x > s) \\
t_{21}(e^{-j\varphi x} + r_{21} e^{j\varphi x} e^{-j(\varphi x + \beta_1^2 x)}) & (-d \geq x).
\end{cases}
\]

\[
k_1 = \sqrt{\eta_1 k_2^2 - \beta_1^2}, \quad k_2 = \sqrt{\eta_2 k_3^2 - \beta_1^2} \quad \text{wavevector components along the x-direction}
\]

\[
t_0 = 2 k_1 / (k_1 + k_2) \quad \text{transmission coefficient from the regions i to j}
\]

\[
r_0 = (k_1 - k_2) / (k_1 + k_2) \quad \text{reflection coefficient from the regions i to j}
\]
Semi-analytical calculation of dispersion relations

For the 1D symmetric lattice,

\[
\epsilon(z) = \sum_{n} \epsilon_{n} \cos(nKz),
\]

\[
\epsilon_{0} = \epsilon_{\text{avg}} = \epsilon_{l} + \rho(\epsilon_{h} - \epsilon_{l}) = (2\Delta\epsilon/\pi)\sin(n\pi\rho).
\]

The size of the 2\textsuperscript{nd}-order band gap is

\[
\text{Re}(|\Omega'' - \Omega'|) = 2|h_{2} - \text{Im}(h_{1})|/(K\hbar_{0})
\]

Note that the sign of \(h_{2}\),

\[
h_{2} = \frac{K\Omega^{2}\epsilon_{2}}{4} \int_{-d}^{0} \psi(x)\psi^{*}(x)dx
\]

When \(\rho > 0.5\) \(\rightarrow\) \(\epsilon_{2} < 0\) \(\rightarrow\) \(h_{2} < 0\) \(\rightarrow\) can be wide-gap opening

When \(\rho < 0.5\) \(\rightarrow\) \(\epsilon_{2} > 0\) \(\rightarrow\) \(h_{2} > 0\) \(\rightarrow\) can be gap closing

At the 2\textsuperscript{nd}-order band gap closure with

\[
h_{2} = \text{Im}(h_{1})
\]

\[
\Omega(k_{z}) = \Omega_{0} - (i\hbar_{1} \pm \sqrt{k_{z}^{2} - \text{Re}(h_{1})^{2}})/(K\hbar_{0}).
\]

\(\rightarrow\) The frequency is fully degenerate (not coalescent) at

\[
k_{\text{ex}} = \pm \text{Re}(h_{1}).
\]

\(\rightarrow\) Exceptional point
Symmetry Breaking in Photonic Crystals: On-Demand Dispersion from Flatband to Dirac Cones
H. S. Nguyen …... Université de Lyon

The eigenvalue $\alpha(k_x)$ of $H$ is
$$\det(H - I) = 0$$
$$\begin{align*}
\Omega_{1,2}^2 &= \alpha_{1,2} - \omega \pm \nu_{1,2} k_x \\
\Omega_{2,1} &= -\omega_{1,2} \pm \nu_{1,2} k_x \\
\Omega_{2,2} &= \omega_{1,2} \pm \nu_{1,2} k_x \\
\Omega_{1,1} &= -\omega_{1,2} \pm \nu_{1,2} k_x
\end{align*}$$

Four eigenstates $|n\rangle$ with $n=1,2,3$ and $4$.

Four eigenstates $|n\rangle$ with $n=1,2,3$ and $4$.

Fundamental zero-order waves of the two noncorrugated waveguide structures:

Consider the first stopband:

1. First-order diffractive coupling processes between backward and forward wave components are most effective.
2. Second-order terms (anti-diagonal terms) that can be neglected.

From the four Fundamental zero-order waves, $|a_{1+, a_{1-}, a_{2+}, a_{2-}}\rangle$.

The weights of even and odd component in each eigenstate are

\[ a_{even} = \left| \langle a_{a_{even}} | a_{even} \rangle \right|^2 \]
\[ a_{odd} = \left| \langle a_{a_{odd}} | a_{odd} \rangle \right|^2 \]

where

\[ \alpha_{even} = \frac{a_{1+} + a_{2-}}{\sqrt{2}} \]
\[ \alpha_{odd} = \frac{a_{1-} - a_{2+}}{\sqrt{2}} \]

(1) Fishbone structure

A double-period design perturbation with a perturbation magnitude $\alpha$

\[ \alpha \ll 1 \]

Since the two sub-gratings are identical,

\[ a_1 = a_2 = a_3, v_1 = v_2 = v_3, \beta_1 = \beta_2 = \beta \]

\[ W_1 = \rho e^{j\psi_1}, W_2 = \rho e^{j\psi_2} \]

\[ \rho = U \sqrt{1 + \beta^2 + 2 \cos \phi} \]

\[ \sin \psi_1 = \frac{\sin \phi}{\sqrt{1 + \beta^2 + 2 \cos \phi}} \]
\[ \sin \psi_2 = \frac{\sin \phi}{\sqrt{1 + \beta^2 + 2 \cos \phi}} \]

\[ \psi_1 - \psi_2 = 0 \]
\[ \psi_1 - \psi_2 = \pi \]

At $k_x=0$,

\[ E_{1+}^0 = \alpha_1 - \sqrt{\nu_1^2 + \rho^2 - \frac{\sqrt{2\left[1 + \cos(\psi_1 - \psi_2)\right]}}{\rho}} \]
\[ E_{2+}^0 = \alpha_2 - \sqrt{\nu_2^2 + \rho^2 + \frac{\sqrt{2\left[1 + \cos(\psi_1 - \psi_2)\right]}}{\rho}} \]
\[ E_{1-}^0 = \alpha_1 + \sqrt{\nu_1^2 + \rho^2 - \frac{\sqrt{2\left[1 + \cos(\psi_1 - \psi_2)\right]}}{\rho}} \]
\[ E_{2-}^0 = \alpha_2 + \sqrt{\nu_2^2 + \rho^2 + \frac{\sqrt{2\left[1 + \cos(\psi_1 - \psi_2)\right]}}{\rho}} \]
Symmetry Breaking in Photonic Crystals: On-Demand Dispersion from Flatband to Dirac Cones

H. S. Nguyen, Université de Lyon

(1) Fishbone structure

Dirac cones

\[ \omega(k_x) - \omega(k_y) = \left( V_f^2 + \rho^2 + \sqrt{V_f^2 + \rho^2} \right) \pm 2 \rho V_f \sqrt{1 + \cos(\psi_1 - \psi_2)} \]

\[ \psi_1 - \psi_2 = \pi \text{ if } \phi = \pi \]

The formation of Dirac cones are not accidental (i.e. independent on \(v_0\), \(U\), \(\beta\) and \(V_f\), thus independent of the geometry and material of the sub-gratings). The only condition is the \(\pi\) phase-shift, corresponding to an offset \(\delta=0.5\).

At the Dirac points, the weight ratio between the odd and even component is

\[ \alpha_{\text{even}} = \frac{U(1-\beta)/V_f}{\sqrt{1+U^2(1-\beta)^2}/V_f^2 + 1} \]

for \(U/V_f \gg 1\), even and odd components are equally balanced (i.e. fully hybridized) for each Dirac point.

(2) Comb structure

the lower sub-grating of the Fishbone is non-corrugated, (i.e. \(U=0\)).

At \(k_x=0\),

\[ E_1^{(0)} = \frac{1}{2} (1+\beta) U \left[ (\alpha_0 - \alpha_1) - U(1-\beta) \right] \]

\[ E_2^{(0)} = \frac{1}{2} (1+\beta) U \left[ (\alpha_0 + \alpha_1) - U(1-\beta) \right] \]

\[ E_3^{(0)} = \frac{1}{2} (1+\beta) U \left[ (\alpha_0 - \alpha_1) + U(1-\beta) \right] \]

\[ E_4^{(0)} = \frac{1}{2} (1+\beta) U \left[ (\alpha_0 + \alpha_1) + U(1-\beta) \right] \]

Replacing the base \([\alpha_0, \alpha_1, \alpha_{2e}, \alpha_{2o}]\) with \([\alpha_{0e}, \alpha_{0o}, \alpha_{2e}, \alpha_{2o}]\)
Reflectivity (TE-pol)

- low index mode (TE$_0$)
- Vertical Even symm
- first excited mode (TE$_1$)
- high index mode (TE$_0$)
- Vertical Even symm

At $\delta/a = 0$, strong coupling between Even (C) and Odd (B)

At $\delta > 0$, vertical symmetry breaking by $\delta$

At $\delta = 0.2$, a flatband at $\Gamma$

At $\delta = 0.5$, Dirac cones at $\Gamma$

(1) Fishbone structure

Flatband at $\Gamma$

Dirac cone at $\Gamma$

At $\delta = 0.1$

($n_{\text{low}} = 1.5$)

($n_{\text{high}} = 3.15$)

$h = 0.21$ $\mu$m

$a = 0.35$ $\mu$m

At $\delta = 0.2$, flatband at $\Gamma$

At $\delta = 0.5$, Dirac cones at $\Gamma$

(참고: coupling TE0, TE1; and GMR modes)

At $\delta = 0.5$

At $\delta = 0.2$

At $\delta = 0.3$

At $\delta = 0.4$

At $\delta = 0.5$
Symmetry Breaking in Photonic Crystals: On-Demand Dispersion from Flatband to Dirac Cones

H. S. Nguyen, Université de Lyon

(이기영 계산) Fishbone structure

Reflectivity (TE-pol)
Symmetry Breaking in Photonic Crystals: On-Demand Dispersion from Flatband to Dirac Cones

H. S. Nguyen …., Université de Lyon

(2) Comb structure

A double-period design perturbation approach with a perturbation magnitude \( \alpha \)

\[ \alpha \ll 1. \]

\[ \begin{align*}
\alpha & \leq 0.5, \\
\text{Dirac cones at } & \Gamma.
\end{align*} \]

Flatband at \( \Gamma \)

- the integration of active materials in these designs, for example, by placing quantum wells in the noncorrugated part of a comb structure, or by depositing monolayers of 2D material on top of the structure via exfoliation.
- the active layer will be unpatterned—an ideal configuration to obtain the strong coupling regime with fragile 2D materials
- the possibility to use multivalley dispersion in the strong coupling regime to obtain spontaneous momentum symmetry breaking and two-mode squeezing, as well as the observation of the Josephson effect in momentum space
Solitons in one-dimensional three-band model with a central flat band
Gyungchoon Go, Kyeong Tae Kang, and Jung Hoon Han, Sungkyunkwan University

천상모 교수님 ???
Topological interface modes in 1-dim. photonic lattices

- Surface impedance
- Zak phase (1-dim. Berry phase)
Impedances and Zak phases

Impedance $Z_{A,B}$ ($\eta_i$) of materials

Surface impedance $Z_{SR}$ of the semi-infinite PC

Inside a band gap, $Z_{sr}$ is a pure imaginary number

The condition for the presence of an interface state

Let us now put another PC,

- the surface impedances on the two sides are opposite in sign
- The sign of the surface impedance for frequencies inside a band gap is, in fact, determined by the geometrical phase (Zak phase) of the bulk bands.

Let’s derive a relation between the surface impedance and the Zak phase
The periodic-in-cell part of the Bloch E field of a state on the \( n \)th band with wave vector \( q \)

\[
E_{x,n,q}(z) = u_{n,q}(z) \exp(iqz)
\]

The 1D system with inversion symmetry always has two inversion centers (A or B)

\( \rightarrow \) the Zak phase is quantized at either 0 or \( \pi \), if the origin is chosen to be one of the inversion center.

**ZAK PHASE = 0 or \( \pi \)**

Show that if one isolated band (excluding the 0th band) contains the frequency point at which (if we set the origin of the system at the center of slab A).

\[
\sin(\tilde{\omega}n_b d_b / c) = 0,
\]

\( \rightarrow \) the Zak phase of this band must be \( \pi \).

\( \rightarrow \) Otherwise, the Zak phase of this band must be 0.

When \( \epsilon_a \) is decreased from 4, we need to increase the value of \( d_a \) in order to keep \( n_a d_a + n_b d_b \) unchanged. Thus, the value of \( n_b d_b \) is reduced accordingly, which in turn implies

\( \tilde{\omega} > \omega_7 \)
C. Zak phase and the symmetry properties of the edge states

As an example, let us focus on the 6th and 7th bands in which the Zak phases change by $\pi$.

- Before crossing, Odd functions $E(z)$ are mapped to Even functions after the band crossing.
- After crossing, Odd functions $E(z)$ become Even functions.
- Even functions $E(z)$ remain unaffected by the band crossing.
- (N, R) Same symmetry, $\rightarrow$ Zak phase = 0
- (Q, R) Different symmetry, $\rightarrow$ Zak phase = $\pi$
- (M, N) Same symmetry, $\rightarrow$ Zak phase = 0
- (L, P) After the band crossing, Odd functions $E(z) \rightarrow$ Even Function
- (M, Q) After the band crossing, Even functions $E(z) \rightarrow$ Odd Function

$\varepsilon_a = 3.8, \varepsilon_b = 1$
$d_a = 0.42\Lambda, d_b = 0.58\Lambda$
$d_a = 0.42\Lambda, d_b = 0.58\Lambda$
$d_a = 0.38\Lambda, d_b = 0.62\Lambda$
$d_a = 0.38\Lambda, d_b = 0.62\Lambda$
D. The sign of impedance and the symmetry properties of the edge states

For (L & Q) Antisymmetric (A) state
- \( E = 0 \) at the boundary
- A perfect electric conductor-like boundary condition
- Reflection must be \( r = -1 = \exp(i\phi) \Rightarrow \phi = \pi \)

For a gap with state A at the lower edge (point Q),
The function \( \varsigma \) has a value 0 at the lower edge and decreases monotonically to \(-\infty\) as the upper edge is approached.

\[ \text{Negative when the lower edge is type A.} \]
\[ \varsigma < 0 \text{ gap} \]

For (M & P) Symmetric (S) state
- \( E = \text{maximum} \) at the boundary
- A perfect magnetic conductor-like boundary condition
- Reflection must be \( r = 1 = \exp(i\phi) \Rightarrow \phi = 0, 2\pi \)

For a gap with state S at the lower edge (point M),
The function \( \varsigma \) has a value \( \infty \) at the lower edge and decreases monotonically to 0 as the upper edge is approached.

\[ \text{Positive when the lower edge is type S.} \]
\[ \varsigma > 0 \text{ gap} \]

The condition for interface states
\[ Z_{SR} + Z_{SL} = 0 \]

If two states at the lower edges of the common gap have different types, an interface state must exist inside the gap.
Zak phase and the sign of impedance

A relation between the surface impedance of the PC in the $n$th gap and the sum of Zak phases of all bands below the $n$th gap:

$$\text{sgn}[\zeta(n)] = (-1)^n (-1)^l \exp \left( \sum_{m=0}^{n-1} \theta_{m}^{Zak} \right)$$

where $i$ is the number of crossing points under the $n$th gap.

$$Z_s^{(n)}/Z_0 = i_s^{(n)}$$

$$\theta_{n}^{Zak} = \int_{-\pi/\Lambda}^{\pi/\Lambda} \left[ \int_{\text{unit cell}} dz \epsilon(z) u_{n,q}(z) \partial_q u_{n,q}(z) \right] dq$$

$$\exp(i\theta_0^{Zak}) = \text{sgn}[1 - \epsilon_a \mu_b / (\epsilon_b \mu_a)] \quad (m = 0)$$

Before crossing

$\zeta > 0$, $\left\langle n = 8, l = 1, \text{Sum}_{\text{Zak}} = 4\pi \right\rangle$

$n = 7$, $l = 0$, $Z = 3\pi$

$\epsilon_a = 3.8$, $\epsilon_b = 1$

$d_a = 0.42\Lambda$, $d_b = 0.58\Lambda$

After crossing

$\zeta < 0$, $\left\langle n = 6, l = 0, \text{Sum}_{\text{Zak}} = 3\pi \right\rangle$

$\zeta > 0$, $\left\langle n = 5, l = 0, \text{Sum}_{\text{Zak}} = 3\pi \right\rangle$

$\zeta > 0$, $\left\langle n = 4, l = 0, \text{Sum}_{\text{Zak}} = 2\pi \right\rangle$

$\zeta < 0$, $\left\langle n = 3, l = 0, \text{Sum}_{\text{Zak}} = 2\pi \right\rangle$

$\zeta < 0$, $\left\langle n = 2, l = 0, \text{Sum}_{\text{Zak}} = \pi \right\rangle$

$\zeta > 0$, $\left\langle n = 1, l = 0, \text{Sum}_{\text{Zak}} = \pi \right\rangle$

$\epsilon_a = 4.2$, $\epsilon_b = 1$

$d_a = 0.38\Lambda$, $d_b = 0.62\Lambda$
Changing the sign of impedance by passing a topological transition point (Dirac point)

\[ Z_{SR} = \frac{1 + r_R Z_0}{1 - r_R} \equiv i \zeta_R \]

The gap with (magenta) \( \zeta > 0 \),

The gap with (cyan) \( \zeta < 0 \).

The gap with (magenta) \( \zeta > 0 \),

The gap with (cyan) \( \zeta < 0 \).

Keep the condition of \( \gamma = k_a d_a + k_b d_b = m\pi \),

Sign change in the surface impedance in the 7th gap

Switching the Zak phase in passbands 6 and 7 (0, \( \pi \))

Topological phase transition arising from band crossing in photonic systems
**A. Band crossing condition**

**Band dispersion relation**

\[
\cos(q\Lambda) = \cos k_d a \cos k_b b = \frac{1}{2} \left( \frac{z_a}{z_b} + \frac{z_b}{z_a} \right) \sin k_d a \sin k_b b,
\]

where \( k_i = \omega n_i / c \), \( n_i = \sqrt{\varepsilon_i} \), \( \varepsilon_i = \sqrt{\mu_i} \), \( (i = a \text{ or } b) \);

\[
\alpha = n_a d_a / (n_b d_b) \rightarrow \text{the ratio of the optical path in A and B}
\]

\[
\gamma = (n_a d_a + n_b d_b) / c = k_a d_a + k_b d_b \rightarrow \text{the phase delay in A+B}
\]

\[
\tau = (z_a / z_b + z_b / z_a) / 2 > 1 \rightarrow \text{the impedance mismatch}
\]

\[
\cos(q\Lambda) = \cos \gamma (1 - \tau) \sin k_d a \sin k_b b
\]

**The sufficient condition for two bands to cross** (either at zone center or zone boundary)

When \( \sin k_i d_i = 0 \), \( \omega = l m_2 \pi c / (n_b d_b) \), \( (l, m_2: \text{integer}) \)

\[
\cos(k_i d_i) = (-1)^l m_2
\]

which is always less than 1

\rightarrow The frequency at which \( \sin(k_i d_i) = 0 \) must be in the pass band.

When \( \alpha = n_a d_a / (n_b d_b) = m_1 / m_2 \), \( (m_1, m_2: \text{integer}) \)

\[
\cos(k_a d_a) = (-1)^l m_1
\]

\[
\sin k_a d_a = 0 \rightarrow \omega = l m_1 \pi c / (n_a d_a)
\]

\[
\cos(q\Lambda) = (-1)^l (m_1 + m_2)
\]

\rightarrow \( q = 0 \) when \( l(m_1 + m_2) \) is even,

\rightarrow \( q = \pm \pi / \Lambda \) when \( l(m_1 + m_2) \) is odd.

Near these frequencies, \( \omega = l m_2 \pi c / (n_b d_b) = l m_1 \pi c / (n_a d_a) \)

The band has linear dispersion. \rightarrow Proof

\rightarrow the degeneracy band point at \( \omega = l m_2 \pi c / (n_b d_b) = l m_1 \pi c / (n_a d_a) \)

\rightarrow \((q_0, \omega_0) = [0, m_2 \pi c / (n_b d_b)] \text{ or } (q_0, \omega_0) = [\pm \pi / \Lambda, m_2 \pi c / (n_b d_b)] \)

\rightarrow Suppose that \((q_1, \omega_1)\) is another band point NEAR \((q_0, \omega_0)\)

\rightarrow \(|q_1 - q_0|, |\omega_1 - \omega_0|\) are small numbers.

\rightarrow Keeping to the lowest order of expansion of the band dispersion,

\[
|q_1 - q_0| = \sqrt{C_1} |\omega_1 - \omega_0| / c \quad C_1 = \left[ (n_a d_a)^2 + (n_b d_b)^2 + \left( \frac{z_a}{z_b} + \frac{z_b}{z_a} \right) n_a d_a n_b d_b \right] / \Lambda^2
\]

(DIRAC POINTS)

\rightarrow When \( \alpha = n_a d_a / (n_b d_b) = m_1 / m_2 \),

\rightarrow bands will cross with linear dispersion.
A. Band crossing condition

The necessary condition for two bands to cross

(1) The cross points of two bands could occur only at the boundary or the center of the BZ for 1D PC cases.

- $q = 0$ when $l(m_1 + m_2)$ is even,
- $q = \pm \pi / \Lambda$ when $l(m_1 + m_2)$ is odd.

- If two bands cross at points other than the center or boundary of the BZ, each frequency would have four corresponding Bloch vectors $q$.

- This is not possible because there can be at most two values of $q$ for each frequency.

(2) If one frequency satisfies the condition $\gamma = \frac{k_ad_a + k_bd_b}{\omega / c} = m \pi$,

- The frequency must be in the band gap if two bands do not cross.

- If two bands cross at $\gamma = m \pi$,

- $\sin k_ad_a = 0$ and $\sin k_bd_b = 0$,

- $\alpha = \frac{k_ad_a}{k_bd_b} = \frac{m_1}{m_2}$

- $\gamma = (n_ad_a + n_bd_b) \omega / c = m \pi$ \(\Rightarrow\) $\omega_m = m \pi c / (n_ad_a + n_bd_b)$

: Mid-gap frequency (or, Crossing frequency)
Simultaneous multi-frequency topological edge modes between one-dimensional photonic crystals

If the $n_A$ of $X$ is tuned just before topological phase transition, and the $n_C$ of $Y$ is just after the transition, the combined structure (Fig. 1) will support a topological edge mode within this band crossing gap due to their $\pi$ difference in

The band crossing condition $\rightarrow$ $(n_A d_A)/(n_B d_B) = s_1/s_2$, where $s_1$ and $s_2$ are integers, band crossing occurs at the $(s_1 + s_2)$th bandgap

- the case of $s_1 = s_2$ (i.e., $n_A d_A = n_B d_B$) $s_1 + s_2$ is always an even number

$\rightarrow$ all the crossings will occur at every bandgap that opens at zone center.
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There are strongly localized fields near the interface between X and Y’. The enhancement at the edge mode frequencies in Fig. 4 are about 20 to 30, while the enhancement region has a two-dimensional (2-D) surface area of beam size of the incident light in plane wave approximation.

The enhancement can increase exponentially with increasing number of units. This result suggests that the interface region is suitable for inserting single or few layers of thin or 2-D materials for enhancing frequency upconversion optical processes.
Using a reflection spectrum measurement, we determined the existence of interface states in the gaps and then obtained the Zak phases.

- If the two band edge states of the same band have the same symmetry, the Zak phase of this band is 0.
- Otherwise, the Zak phase is $\pi$.

The determination of the symmetry types of the band edge states is quite difficult to implement experimentally.

The Zak phase is related to the signs of the reflection phases, the reflection phases of the $(n-1)$th gap and $n$th gap

$$\exp\left(i\theta_n^{Zak}\right) = -\text{sgn}(\varphi_n)/\text{sgn}(\varphi_{n-1}),$$

$\Rightarrow$ we only need the sign, which greatly simplifies the measurement.

The existence condition of an interface state is

$$\varphi_{PC} + \varphi_{Ag} = 0.$$
A topological PhC nanocavity with a near-diffraction-limited mode volume and its application to single-mode lasing. The topological origin of the nanocavity, formed at the interface between two topologically distinct PhCs, guarantees the existence of only one mode within its photonic bandgap.