Chapter 2. 
Geometrical Optics

Light Ray: the path along which light energy is transmitted from one point to another in an optical system.

Speed of Light: Speed of light (in vacuum): a fundamental (or a “defined”) constant of nature given by
\[ c = 299,792,458 \text{ meters / second} = 186,300 \text{ miles / second}. \]

Index of Refraction

\[ n = \frac{c}{v}, \]
where \( c \) = speed of light in vacuum and \( v \) = speed of light in the material.

Some examples of various types of matter and the associated index of refraction include:

<table>
<thead>
<tr>
<th>matter</th>
<th>index of refraction ( n )</th>
<th>velocity of light ( v )</th>
</tr>
</thead>
<tbody>
<tr>
<td>absolute vacuum</td>
<td>( n = 1 )</td>
<td>( v = c )</td>
</tr>
<tr>
<td>air</td>
<td>( n = 1.0003 )</td>
<td>( v = 0.9997 c )</td>
</tr>
<tr>
<td>water</td>
<td>( n = 1.33 )</td>
<td>( v = 0.75 c )</td>
</tr>
<tr>
<td>glass</td>
<td>( 1.4 &lt; n &lt; 1.8 )</td>
<td>( 0.56 c \leq v \leq 0.71 c )</td>
</tr>
<tr>
<td>diamond</td>
<td>( n = 2.4 )</td>
<td>( v = 0.42 c )</td>
</tr>
<tr>
<td>silicon</td>
<td>( n = 3.5 )</td>
<td>( v = 0.29 c )</td>
</tr>
</tbody>
</table>
The optical path length in a medium is the integral of the refractive index and a differential geometric length:

\[
OPL = \int_a^b n \, ds
\]
Reflection and Refraction

Plane contains surface normal, incident, reflected, and refracted rays.

Reflection: $\theta_r = \theta_i$

Refraction: $\frac{\sin \theta_i}{\sin \theta_r} = \frac{n_t}{n_i} = \text{constant}$
Plane of incidence
Fermat’s Principle: Law of Reflection

Fermat’s principle:
Light rays will travel from point A to point B in a medium along a path that minimizes the time of propagation.

\[ OPL_{AB} = n \sqrt{(-x_1)^2 + (y_2 - y_1)^2} + n \sqrt{(-x_2)^2 + (y_3 - y_2)^2} \]

Fix \( x_1, y_1, x_3, y_3 \)

\[ \frac{dOPL_{AB}}{dy_2} = 0 = \frac{n \frac{1}{2} 2(y_2 - y_1)}{\sqrt{(x_1)^2 + (y_2 - y_1)^2}} + \frac{n \frac{1}{2} 2(y_3 - y_2)}{\sqrt{(x_3)^2 + (y_3 - y_2)^2}} \]

0 = \frac{n(y_2 - y_1)}{\sqrt{(x_1)^2 + (y_2 - y_1)^2}} - \frac{n(y_3 - y_2)}{\sqrt{(x_3)^2 + (y_3 - y_2)^2}}

0 = n \sin \theta_i - n \sin \theta_r

\[ \sin \theta_i = \sin \theta_r \]

\[ \theta_i = \theta_r \quad \text{: Law of reflection} \]
Fermat’s Principle: Law of Refraction

Law of refraction:

\[ OPL_{AB} = n_i \sqrt{(x_2-x_1)^2 + (y_1)^2} + n_t \sqrt{(x_3-x_2)^2 + (-y_3)^2} \]

Fix \( x_1, y_1, x_3, y_3 \)

\[
\frac{d(OPL_{AB})}{dy_2} = 0 = \frac{n_i \frac{1}{2} 2(x_2-x_1)}{\sqrt{(x_2-x_1)^2 + (y_1)^2}} + \frac{n_t \frac{1}{2} 2(x_3-x_2)(-1)}{\sqrt{(x_3-x_2)^2 + (y_3)^2}}
\]

\[
0 = \frac{n_i (x_2-x_1)}{\sqrt{(x_2-x_1)^2 + (y_1)^2}} - \frac{n_t (x_3-x_2)}{\sqrt{(x_3-x_2)^2 + (y_3)^2}}
\]

\[ 0 = n_i \sin \theta_i - n_t \sin \theta_t \]

\[ \Rightarrow n_i \sin \theta_i = n_t \sin \theta_t \]

\[ n_i \theta_i = n_t \theta_t \]: Law of refraction in paraxial approx.
Refraction – Snell’s Law: \( n_i \sin \theta_i = n_t \sin \theta_t \)

\[ n_i < n_t \Rightarrow \theta_i > \theta_t \]

\[ n_i > n_t \Rightarrow \theta_i < \theta_t \]

\[ n_i \times n_t < 0 \]
Negative index of refraction: $n < 0$

$$k = \omega \sqrt{\varepsilon \mu} = k_0 n, \quad n = \sqrt{\varepsilon \mu},$$

$$v = \frac{c}{\sqrt{\varepsilon \mu} n} = \frac{c}{\sqrt{\varepsilon \mu} n},$$
If Point B is the source of light rays, Fermat’s principle must predict the same path as determined for the original direction of light propagation.

**Principle of Reversibility**

*Any actual ray of light in an optical system, if reversed in direction, will retrace the same path backward.*

(A simple but useful principle.)
2-4. Reflection in plane mirrors

Describe the direction of the light ray by its unit vector.

Consider reflection from one surface:
\[ \hat{\mathbf{r}}_{AB} = (x, y, z) \rightarrow \hat{\mathbf{r}}_{BC} = (x, y, -z) \]

Now consider reflection sequentially from all three rectangular coordinate planes ("Corner reflector"):
\[ \hat{\mathbf{r}}_{AB} = (x, y, z) \rightarrow \hat{\mathbf{r}}_{DE} = (-x, -y, -z) \]

→ The ray returns precisely parallel to the line of its original approach.

Example in our life:
Plane surface – Image formation

A point

\[ \overrightarrow{SS'} \perp \overrightarrow{NP} \]
\[ \overrightarrow{SN} = \overrightarrow{S'N} \]

A line

- Upside down
- Transverse orientation of object and image are the same
- Unity magnification

A 3-D object

Two reflecting surfaces

These are all virtual image (the image cannot be projected on a screen as in the case of a real image).
Total internal Reflection (TIR)

\[ \theta_c = \sin^{-1}\left(\frac{n_2}{n_1}\right) \]

Example 1: Diamond

Most of the rays entering the top of the diamond will exit from the top due to total internal reflection.

Example 2: Optical Fiber

\[ n_{\text{core}} > n_{\text{cladding}} \]
2-6. Imaging by an Optical System

Wavefront: Spherical surfaces normal to the light rays

- **Fermat’s principle** → All optical paths result in the same travel time.

- **Principle of reversibility** applies.

Points O and I are called *conjugate points*. 
Cartesian Surfaces

• A Cartesian surface – those which form perfect images of a point object
• E.g. ellipsoid and hyperboloid
Imaging by Cartesian reflecting surfaces

- **Plane mirror**
- **Elliptical mirror**
  - The optical paths obey Hero’s principle.
  - → $P_1$ and $P_2$ are the two foci.
- **Hyperbolic mirror**
- **Parabolic mirror**

**Quiz:** Which ones are real images? Which ones are virtual images?

**Example in real life:**
Imaging by Cartesian refracting Surfaces

Ellipsoid surface

$n_o > n_i$

Focus

Hyperbolic surface

$n_o < n_i$

Focus

Double-hyperbolic lens

Aberration-free imaging
The optical path length for any path from Point O to the image Point I must be the same by Fermat’s principle.

\[ n_o d_o + n_i d_i = n_o s_o + n_i s_i = \text{constant} \]

\[ n_o \sqrt{x^2 + y^2 + z^2} + n_i \sqrt{(s_o + s_i - x)^2 + y^2 + z^2} = n_o s_o + n_i s_i = \text{constant} \]

The Cartesian or perfect imaging surface is a paraboloid in three dimensions. Usually, though, lenses have spherical surfaces because they are much easier to manufacture.

→ Spherical approximation or, paraxial approx.
Approximation by Spherical Surfaces

Hyperbolic surface

Parabolic mirror

Spherical aberration

Paraxial approximation

But spherical surface is easier to make!
(We cannot have everything.)
2-7. Reflection at a Spherical Surface

Paraxial approximation: \( \theta \) is small, \( \sin \theta \approx \tan \theta \approx \theta \)

\[
\frac{1}{s} + \frac{1}{s'} = \frac{1}{f}
\]

\[
f = \begin{cases} 
R & > 0 \quad \text{(Concave mirror, } R < 0) \\
\frac{R}{2} & < 0 \quad \text{(Convex mirror, } R > 0) 
\end{cases}
\]

\( s' \) is positive for real image, negative for virtual image
Reflection from a spherical convex surface gives rise to a virtual image. Rays appear to emanate from point I behind the spherical reflector.

Use paraxial or small-angle approximation for analysis of optical systems:

\[
\sin \varphi = \varphi - \frac{\varphi^3}{3!} + \frac{\varphi^5}{5!} - \cdots \approx \varphi
\]

\[
\cos \varphi = 1 - \frac{\varphi^2}{2!} + \frac{\varphi^4}{4!} - \cdots \approx 1
\]
Considering Triangle OPC and then Triangle OPI we obtain:

\[ \theta = \alpha + \varphi \quad 2\theta = \alpha + \alpha' \]

Combining these relations we obtain:

\[ \alpha - \alpha' = -2\varphi \]

Again using the small angle approximation:

\[ \alpha \approx \tan \alpha \approx \frac{h}{s} \quad \alpha' \approx \tan \alpha' \approx \frac{h}{s'} \quad \varphi \approx \tan \varphi \approx \frac{h}{R} \]
Reflection at Spherical Surfaces III

Image distance $s'$ in terms of the object distance $s$ and mirror radius $R$:

$$\frac{h}{s} - \frac{h}{s'} = -2\frac{h}{R} \quad \Rightarrow \quad \frac{1}{s} - \frac{1}{s'} = -\frac{2}{R}$$

At this point the sign convention in the book is changed!

$$\frac{1}{s} + \frac{1}{s'} = -\frac{2}{R}$$

The following sign convention must be followed in using this equation:

1. Assume that light propagates from left to right. Object distance $s$ is positive when point O is to the left of point V.

2. Image distance $s'$ is positive when I is to the left of V (real image) and negative when to the right of V (virtual image).

3. Mirror radius of curvature $R$ is positive for C to the right of V (convex), negative for C to left of V (concave).
The focal length $f$ of the spherical mirror surface is defined as $-R/2$, where $R$ is the radius of curvature of the mirror. In accordance with the sign convention of the previous page, $f > 0$ for a concave mirror and $f < 0$ for a convex mirror. The imaging equation for the spherical mirror can be rewritten as

$$\frac{1}{s} + \frac{1}{s'} = \frac{1}{f}$$
Reflection at Spherical Surfaces V

Ray 1: Enters from O' through C, leaves along same path
Ray 2: Enters from O' through F, leaves parallel to optical axis
Ray 3: Enters through O' parallel to optical axis, leaves along line through F and intersection of ray with mirror surface

\[ s = 7 \text{ cm} \quad R = +8 \text{ cm} \quad f = -R/2 \quad \frac{1}{s'} = \frac{1}{f} - \frac{1}{s} \]

\[ s' = ? \quad m = -\frac{s'}{s} = ? \]
Reflection at Spherical Surfaces VI

\[ s = +17 \text{ cm} \quad R = -8 \text{ cm} \quad f = -R/2 = \]

\[ \frac{1}{s'} = \frac{1}{f} - \frac{1}{s} = \]

\[ s' = \quad m = - \frac{s'}{s} = \]
Reflection at Spherical Surfaces VII

Real, Inverted Image

Virtual Image, Not Inverted

\[ s > f \quad \Rightarrow \quad \frac{1}{s'} = \frac{1}{f} - \frac{1}{s} > 0 \]
\[ m = -\frac{s'}{s} < 0 \]

\[ s < f \quad \Rightarrow \quad \frac{1}{s'} = \frac{1}{f} - \frac{1}{s} < 0 \]
\[ m = -\frac{s'}{s} > 0 \]
At point P we apply the law of refraction to obtain

\[ n_1 \sin \theta_1 = n_2 \sin \theta_2 \]

Using the small angle approximation we obtain

\[ n_1 \theta_1 = n_2 \theta_2 \]

Substituting for the angles \( \theta_1 \) and \( \theta_2 \) we obtain

\[ n_1 (\alpha - \varphi) = n_2 (\alpha' - \varphi) \]

Neglecting the distance QV and writing tangents for the angles gives

\[ n_1 \left( \frac{h}{s} - \frac{h}{R} \right) = n_2 \left( \frac{h}{|s'|} - \frac{h}{R} \right) \]
Rearranging the equation we obtain

\[ \frac{n_1}{s} - \frac{n_2}{|s'|} = \frac{n_1 - n_2}{R} \]

Using the same sign convention as for mirrors we obtain

\[ \frac{n_1}{s} + \frac{n_2}{|s'|} = \frac{n_2 - n_1}{R} = P \]

*P* : power of the refracting surface
Refraction at Spherical Surfaces III

\[ s = 7 \text{ cm} \quad R = +8 \text{ cm} \quad n_1 = 1.0 \quad n_2 = 4.23 \]

\[
\frac{n_1}{s} + \frac{n_2}{s'} = \frac{n_2 - n_1}{R} \quad \Rightarrow \quad s' = ?
\]
Example 2-2: Concept of imaging by a lens

Step 1:
Obtain the image after the first refracting surface.

Step 2:
Treat the image obtained above as a virtual object ($s < 0$), and find out its image after the second refracting surface.
2-9. Thin (refractive) lenses

(a) Converging, positive, or convex lenses

(b) Diverging, negative, or concave lenses
For surface 1:

\[
\frac{n_1}{s_1} + \frac{n_2}{s'_1} = \frac{n_2 - n_1}{R_1}
\]
The Thin Lens Equation II

For surface 1:

\[ \frac{n_1}{s_1} + \frac{n_2}{s'_1} = \frac{n_2 - n_1}{R_1} \]

For surface 2:

\[ \frac{n_2}{s_2} + \frac{n_1}{s'_2} = \frac{n_1 - n_2}{R_2} \]

Object for surface 2 is virtual, with \( s_2 \) given by:

\[ s_2 = t - s'_1 \]

For a thin lens:

\[ t \not\parallel s_2 , s'_1 \quad \Rightarrow \quad s_2 = -s'_1 \]

Substituting this expression we obtain:

\[ \frac{n_1}{s_1} + \frac{n_2}{s'_1} - \frac{n_2}{s'_1} + \frac{n_1}{s_2'} = \frac{n_1}{s_1} + \frac{n_1}{s'_2} = \frac{n_2 - n_1}{R_1} + \frac{n_1 - n_2}{R_2} = P_1 + P_2 \]
Simplifying this expression we obtain:

\[
\frac{1}{s_1} + \frac{1}{s'} = \left(\frac{n_2 - n_1}{n_1}\right) \left(\frac{1}{R_1} - \frac{1}{R_2}\right)
\]

For the thin lens:

\[
s = s_1 \quad s' = s'_2 \quad \Rightarrow \quad \frac{1}{s} + \frac{1}{s'} = \left(\frac{n_2 - n_1}{n_1}\right) \left(\frac{1}{R_1} - \frac{1}{R_2}\right)
\]

The focal length for the thin lens is found by setting \( s = \infty \):

\[
s = \infty \quad \Rightarrow \quad \frac{1}{s'} = \frac{1}{f} = \left(\frac{n_2 - n_1}{n_1}\right) \left(\frac{1}{R_1} - \frac{1}{R_2}\right)
\]
The Thin Lens Equation IV

In terms of the focal length $f$ the thin lens equation becomes:

$$\frac{1}{s} + \frac{1}{s'} = \frac{1}{f}$$

The focal length of a thin lens is positive for a convex lens, negative for a concave lens.
Image Formation by Thin Lenses

Convex Lens

Concave Lens

\[ m = -\frac{s'}{s} \]
Convex Lens, focal length = 5 cm:

\[ \frac{1}{s'} = \frac{1}{f} - \frac{1}{s} \]

\[ f = +5 \text{ cm} \quad s = +9 \text{ cm} \quad \Rightarrow \quad s' = \]

\[ m = -s'/s = \]
Image Formation by Concave Lens

Concave Lens, focal length = -5 cm:

\[ \frac{1}{s'} = \frac{1}{f} - \frac{1}{s} \quad f = -5 \text{ cm} \quad s = +9 \text{ cm} \quad \Rightarrow \quad s' = \]

\[ m = -s'/s = \]
Image Formation: Two-Lens System I

\[ \frac{1}{s_1'} = \frac{1}{f_1} - \frac{1}{s_1} = \frac{s_1 - f_1}{s_1 f_1} \quad f_1 = +15 \text{ cm} \quad s_1 = +25 \text{ cm} \Rightarrow s_1' = \]

\[ \frac{1}{s_2'} = \frac{1}{f_2} - \frac{1}{s_2} = \frac{s_2 - f_2}{s_2 f_2} \quad f_2 = -15 \text{ cm} \quad s_2 = \Rightarrow s_2' = \]

\[ m = m_1 m_2 = \]
\[
\frac{1}{s'_1} = \frac{1}{f_1} - \frac{1}{s_1} \quad f_1 = +3.5 \text{ cm} \quad s_1 = +5.2 \text{ cm} \quad \Rightarrow \quad s'_1 = \\
\frac{1}{s'_2} = \frac{1}{f_2} - \frac{1}{s_2} \quad f_2 = +1.8 \text{ cm} \quad s_2 = \quad \Rightarrow \quad s'_2 = \\
\]

\[m = m_1 m_2 = \]
### TABLE 2-1 SUMMARY OF GAUSSIAN MIRROR AND LENS FORMULAS

<table>
<thead>
<tr>
<th></th>
<th>Spherical surface</th>
<th>Plane surface</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reflection</td>
<td>( \frac{\frac{1}{s} + \frac{1}{s'}}{f} = -\frac{R}{2} )</td>
<td>( s' = -s )</td>
</tr>
<tr>
<td></td>
<td>( m = -\frac{s'}{s} )</td>
<td>( m = +1 )</td>
</tr>
<tr>
<td></td>
<td>Concave: ( f &gt; 0, R &lt; 0 )</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Convex: ( f &lt; 0, R &gt; 0 )</td>
<td></td>
</tr>
<tr>
<td>Refraction Single surface</td>
<td>( \frac{n_1}{s} + \frac{n_2}{s'} = \frac{n_2 - n_1}{R} )</td>
<td>( s' = -\frac{n_2}{n_1} s )</td>
</tr>
<tr>
<td></td>
<td>( m = -\frac{n_2 s'}{n_2 s} )</td>
<td>( m = +1 )</td>
</tr>
<tr>
<td></td>
<td>Concave: ( R &lt; 0 )</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Convex: ( R &gt; 0 )</td>
<td></td>
</tr>
<tr>
<td>Refraction Thin lens</td>
<td>( \frac{1}{s} + \frac{1}{s'} = \frac{1}{f} )</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( \frac{1}{f} = \frac{n_2 - n_1}{n_1} \left( \frac{1}{R_1} - \frac{1}{R_2} \right) )</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( m = -\frac{s'}{s} )</td>
<td></td>
</tr>
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<td></td>
<td>Concave: ( f &lt; 0 )</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Convex: ( f &gt; 0 )</td>
<td></td>
</tr>
</tbody>
</table>
Vergence and refractive power: Diopter

\[
\frac{1}{s} + \frac{1}{s'} = \frac{1}{f}
\]

reciprocals

\[
V + V' = P
\]

D > 0

D < 0

Vergence (V): curvature of wavefront at the lens

Refracting power (P)

Diopter (D): unit of vergence (reciprocal length in meter)
Two more useful equations

Stack multiple thin lenses back to back

\[ P = P_1 + P_2 + P_3 + \cdots \quad \leftrightarrow \quad \frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2} + \frac{1}{f_3} + \cdots \]

Newton’s equation for the thin-lens

\[ xx' = f^2 \]
2-12. Cylindrical lenses

(a) Convex

(b) Concave

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Cylindrical lenses

Top view

Side view (b)

Parallel to cylinder axis

Line image