A plane wave

\[ E = E_0 e^{i(-\omega t + \beta z)} \] or \[ E = E_0 e^{i(\omega t - \beta z)} \], the waves are forward waves;

\[ E = E_0 e^{i(\omega t + \beta z)} \] or \[ E = E_0 e^{i(-\omega t - \beta z)} \], the waves are backward waves.

\[ E = \text{Re} \left[ E_0 e^{i(-\omega t + \beta z)} \right] = E_0 \cos(-\omega t + \beta z) \]

The peak has moved to a positive new location at \( \Delta z = (\omega/\beta) \Delta t \).
Thus, this equation represents a forward wave with the phase velocity

\[ v_p = \omega/\beta \]

On the other hand, as time increases with

\[ E = E_0 \cos(\omega t + \beta z) \]

the peak moves toward the negative \( z \) direction, \( \Delta z = -(\omega/\beta) \Delta t \),
and this represents the backward wave.

In this book the convention of \( e^{-j\omega t} \) is used, unless otherwise stated,
because the forward wave \( E = E_0 e^{i(-\omega t + \beta z)} \) has a positive sign on the \( z \).
Complex representation

\[ E(z,t) = |E| \cos(\omega t - kz) = E_o \cos(\omega t - kz) \quad \leftrightarrow \quad E(z,t) = |E| e^{j(\omega t - kz)} = E_o e^{j(\omega t - kz)} \]

We must understand that the complex number so defined is NOT a real physical component since no electromagnetic field in physics is complex; actually, field vector have NO imaginary part, ONLY real part. Using complex representations, the monochromatic fields are written as exponential functions is only for mathematical simplification.

\[
\begin{align*}
a(t) &= |A| e^{j\omega t} \\
&= \text{Re}\{Ae^{j\omega t}\}
\end{align*}
\[
\begin{align*}
b(t) &= |B| e^{j\omega t} \\
&= \text{Re}\{Be^{j\omega t}\}
\end{align*}
\]

with \(A = |A| e^{j\alpha}\) \(B = |B| e^{j\beta}\)

Using the real function:

\[
a(t)b(t) = \text{Re}\{A|B|\cos(\omega t + \alpha)\cos(\omega t + \beta)\} = \frac{1}{2} |A| |B| \left[ \cos(2\omega t + \alpha + \beta) + \cos(\alpha - \beta) \right]
\]

But if we were to evaluate the product \(a(t)b(t)\) with the complex form,

\[
a(t)b(t) = AB e^{j2\omega t} = |A||B| e^{j2\omega t + \alpha + \beta} \quad \rightarrow \quad \text{Re}\{a(t)b(t)\} = |A||B| \cos(2\omega t + \alpha + \beta)
\]

Simpler but... DC term \(\frac{1}{2}AB\cos(\alpha - \beta)\) is missing!

The product of the real part of two complex number may not be equal to the real part of the product of these two complex number! Thus the use of the complex form led to an error!
Consider the time-averaged values which are meaningful, rather than the instantaneous values of many physical quantities. (Since the field vectors are rapidly varying function of time; for example \( \lambda = 1 \ \mu m \) has \( 0.33 \times 10^{-14} \) sec time-varying period!)

\[
\langle a(t)b(t) \rangle = \frac{1}{T} \int_{0}^{T} |A||B| \cos(\omega t + \alpha) \cos(\omega t + \beta) = \frac{1}{2} |AB| \cos(\alpha - \beta)
\]

\[
\langle (a + a^*)(b + b^*) \rangle = \langle ab + ab^* + a^*b + a^*b^* \rangle
\]

Now, it’s identical!!

Keep the complex representation until you reach final answer!!
Optical Intensity and Power
The flow of electromagnetic power is governed by the time average of the Poynting vector \( \mathcal{S} = \mathbf{E} \times \mathbf{H} \). In terms of the complex amplitudes,

\[
\mathcal{S} = \text{Re}\{\mathbf{E}e^{j\omega t}\} \times \text{Re}\{\mathbf{H}e^{j\omega t}\} = \frac{1}{2}(\mathbf{E}e^{j\omega t} + \mathbf{E}^*e^{-j\omega t}) \times \frac{1}{2}(\mathbf{H}e^{j\omega t} + \mathbf{H}^*e^{-j\omega t})
\]

\[
= \frac{1}{4}(\mathbf{E} \times \mathbf{H}^* + \mathbf{E}^* \times \mathbf{H} + \mathbf{E} \times \mathbf{H} e^{2j\omega t} + \mathbf{E}^* \times \mathbf{H}^* e^{-2j\omega t}).
\]

The terms containing \( e^{j2\omega t} \) and \( e^{-j2\omega t} \) are washed out by the averaging process so that

\[
\langle \mathcal{S} \rangle = \frac{1}{4}(\mathbf{E} \times \mathbf{H}^* + \mathbf{E}^* \times \mathbf{H}) = \frac{1}{2}(\mathbf{S} + \mathbf{S}^*) = \text{Re}\{\mathbf{S}\}, \tag{5.3-8}
\]

where

\[
\mathbf{S} = \frac{1}{2}\mathbf{E} \times \mathbf{H}^* \tag{5.3-9}
\]

is regarded as a “complex Poynting vector.” The optical intensity is the magnitude of the vector \( \text{Re}\{\mathbf{S}\} \).
**Nonlinear Wave Equation**

\[
\nabla^2 \mathcal{E} - \frac{1}{c_0^2} \frac{\partial^2 \mathcal{E}}{\partial t^2} = \mu_0 \frac{\partial^2 \mathcal{P}}{\partial t^2}
\]

\[
\mathcal{P} = \varepsilon_0 \chi \mathcal{E} + 2 d \mathcal{E}^2 + 4 \chi^{(3)} \mathcal{E}^3 + \ldots
\]

\[
\mathcal{P} = \varepsilon_0 \chi \mathcal{E} + \mathcal{P}_{NL}, \quad \mathcal{P}_{NL} = 2 d \mathcal{E}^2 + 4 \chi^{(3)} \mathcal{E}^3 + \ldots
\]

Using the relations \( n^2 = 1 + \chi, c_0 = 1/(\mu_0 \varepsilon_0)^{1/2}, \) and \( c = c_0/n, \)

---

**SECOND-ORDER NONLINEAR OPTICS**

\[
\mathcal{P}_{NL} = 2 d \mathcal{E}^2
\]

**Second-Harmonic Generation**

\[
\mathcal{E}(t) = \text{Re}\{E(\omega) \exp(j\omega t)\}
\]

\[
\mathcal{P}_{NL}(t) = P_{NL}(0) + \text{Re}\{P_{NL}(2\omega) \exp(j2\omega t)\},
\]

where \( P_{NL}(0) = d E(\omega) E^*(\omega) \)

\[
P_{NL}(2\omega) = d E(\omega) E(\omega)
\]

---

**THIRD-ORDER NONLINEAR OPTICS**

\[
\mathcal{P}_{NL} = 4 \chi^{(3)} \mathcal{E}^3
\]

**Third-Harmonic Generation**

\[
\mathcal{E}(t) = \text{Re}\{E(\omega) \exp(j\omega t)\}
\]

\[
\mathcal{P}_{NL} = 4 \chi^{(3)} \mathcal{E}^3
\]

\[
\mathcal{P}_{NL}(t) \text{ containing a component at frequency } \omega
\]

and another at frequency \( 3\omega, \)

\[
P_{NL}(\omega) = 3 \chi^{(3)} |E(\omega)|^2 E(\omega)
\]

\[
P_{NL}(3\omega) = \chi^{(3)} E^3(\omega)
\]
Plane waves : 2D

E(x, y, t) : a plane wave propagating at a speed of c in the \( \hat{e} \) direction observed at point P.

\[
\bar{E}(x, y, t) = \bar{E}_r(x, y)e^{-j\omega t} = \bar{E}_0(0,0)e^{-j(\omega t - \vec{k} \cdot \vec{r})} ; \quad \vec{k} = \frac{\omega}{c} \hat{e}
\]

\( \vec{E}_0(0, 0) \) represents the amplitude and direction of polarization

\[
\vec{r} = x\hat{I} + y\hat{J}
\]

\[
\hat{e} = e_x\hat{I} + e_y\hat{J} \quad \text{where} \quad e_x = \cos \theta \quad \text{and} \quad e_y = \sin \theta
\]

\( \vec{k} \) is called the vector propagation constant.

\[
\vec{k} = \frac{\omega}{c} \hat{e} = \frac{2\pi}{\lambda} \hat{e}
\]

\[
k = \frac{2\pi}{\lambda} \cos \theta \hat{I} + \frac{2\pi}{\lambda} \sin \theta \hat{J}
\]

\[
E(x, y, t) = E_0(0, 0)e^{-j\omega t + j\vec{k} \cdot \vec{r}}
\]

Inside a linear medium with refractive index \( n \), the frequency does not change, but the propagation constant becomes \( n\vec{k} \).
Spatial frequency

Spatial frequency $f_s$ is defined as the number of wavelengths in a unit of distance:

$$f_s = \frac{1}{\lambda} : \text{lines/mm or lines/n or cm}^{-1}.$$  

Temporal frequency $f = \frac{c f_s}{\lambda}$

$$\omega = 2\pi f \quad \text{where } f: \text{temporal frequency (1/sec)}$$

$$k = \frac{2\pi}{\lambda} = 2\pi f_s \quad \text{where } f_s = \frac{1}{\lambda} \text{ spatial frequency (1/mm)}$$

$$\vec{k} = \frac{2\pi}{\lambda} \cos \theta \hat{i} + \frac{2\pi}{\lambda} \sin \theta \hat{j} \quad \hat{x}$$

$$\vec{k} = 2\pi f_x \hat{i} + 2\pi f_y \hat{j}$$

$$\mathbf{k} \cdot \mathbf{r} = 2\pi f_x x + 2\pi f_y y$$

$$f_x = \frac{1}{\lambda_x} \quad f_y = \frac{1}{\lambda_y} \quad \rightarrow \quad f_s^2 = f_x^2 + f_y^2$$

$$\lambda_x = \frac{\lambda}{\cos \theta} \quad \lambda_y = \frac{\lambda}{\sin \theta} \quad \rightarrow \quad \left(\frac{1}{\lambda}\right)^2 = \left(\frac{1}{\lambda_x}\right)^2 + \left(\frac{1}{\lambda_y}\right)^2$$

$$f_x = \frac{1}{\lambda_x} \quad f_y = \frac{1}{\lambda_y} \quad \rightarrow \quad f_s^2 = f_x^2 + f_y^2$$
Plane waves: 3D

\[ \hat{e} = \alpha \hat{I} + \beta \hat{J} + \gamma \hat{k} \]

\[ \sqrt{\alpha^2 + \beta^2 + \gamma^2} = 1 \]

\( (\alpha, \beta, \gamma) \ldots \) directional cosine

\[ \alpha = \hat{e} \cdot \hat{I} = \cos \theta \]
\[ \beta = \hat{e} \cdot \hat{J} = \sin \phi \cos \theta \]
\[ \gamma = \hat{e} \cdot \hat{k} = \cos \phi \]

\[ a = \cos^{-1} \alpha \]
\[ b = \cos^{-1} \beta \]
\[ c = \cos^{-1} \gamma \]

\[ k = \frac{2\pi \cos \theta \sin \phi}{\lambda} \hat{I} + \frac{2\pi \sin \theta}{\lambda} \sin \phi \hat{J} + \frac{2\pi \cos \phi}{\lambda} \hat{k} \]

\[ = 2\pi f_x \hat{I} + 2\pi f_y \hat{J} + 2\pi f_z \hat{k} \]

\[ = \frac{2\pi}{\lambda} (\alpha \hat{I} + \beta \hat{J} + \sqrt{1 - \alpha^2 - \beta^2} \hat{k}) \]

\[ \alpha = \lambda f_x \quad \beta = \lambda f_y \quad \gamma = \lambda f_z \]
3D Plane waves: Example 1.2

A plane wave which propagates in a given medium can be expressed as: \( \mathbf{E}(x, y, z, t) = E_0 e^{j(2\pi x + 3y + 4z) \times 10^6 - j10^{15} t} \)

(a) The unit vector of the propagation direction:

\[
\hat{\mathbf{e}} = \frac{1}{\sqrt{29}} (2\hat{\mathbf{I}} + 3\hat{\mathbf{J}} + 4\hat{\mathbf{K}}) = (0.37\hat{\mathbf{I}} + 0.56\hat{\mathbf{J}} + 0.74\hat{\mathbf{K}}) = (\alpha\hat{\mathbf{I}} + \beta\hat{\mathbf{J}} + \gamma\hat{\mathbf{K}})
\]

(b) What’s the value of \( \theta \) and \( \phi \)

\[
\alpha = \sin \phi \cos \theta = 0.37 \quad \beta = \sin \phi \sin \theta = 0.56 \quad \gamma = \cos \theta = 0.74
\]

\( \theta = 56.3^\circ \quad \phi = 42^\circ \)

(c) Find the refractive index of the medium

\[
k \cdot r = 2\pi (f_x x + f_y y + f_z z) = (2x + 3y + 4z) \times 10^6
\]

\[
f_x = \frac{2 \times 10^6}{2\pi} \text{ lines/m} \quad f_y = \frac{3 \times 10^6}{2\pi} \text{ lines/m} \quad f_z = \frac{4 \times 10^6}{2\pi} \text{ lines/m}
\]

\( f_x = 0.86 \times 10^6 \) lines/m

\( \lambda = 1.17 \times 10^{-6} \) m

Temporal frequency \( f = 10^{15}/2\pi = 1.59 \times 10^{14} \text{ Hz} \)

the phase velocity \( v = f \lambda = 1.86 \times 10^{14} \text{ m/sec} \)

The refractive index \( n = \frac{c}{v} = 1.61 \)

(d) Find the vector expression of \( E_0 \), assuming that \( E_0 \) is polarized in the x-plane and the magnitude is 5.

Assume \( E_0 = (a\hat{\mathbf{I}} + b\hat{\mathbf{J}} + c\hat{\mathbf{K}}) \) Since \( E_0 \) is polarized in the x-plane, \( a = 0 \)

the direction of electric field (polarization) is perpendicular to the propagation direction.

\[
E_0 \cdot \hat{\mathbf{e}} = 0
\]

\[
0.56b + 0.74c = 0 \quad \sqrt{b^2 + c^2} = 5 \quad \Rightarrow \quad E_0 = \pm (4\hat{\mathbf{J}} - 3\hat{\mathbf{K}}) \]
Physical meaning of spatial frequency

A typical phase and amplitude distribution of the field diffracted from an aperture source whose dimensions are much smaller than the distance.

Sometimes the angular distribution rather than the planar distribution is desired.

\[ \beta = \lambda f_y \rightarrow f_y = \frac{\cos \theta}{\lambda} = \frac{\sin \phi}{\lambda} \rightarrow \sin \phi = \lambda f_y \]

Therefore, diffracted angle corresponds to the spatial frequency!

Mathematically, \( \lambda_{y_i} \) at the observation point \( P(y_i, z_i) \) is,

\[ \lambda_{y_i} = \frac{\lambda}{\sin \phi} \rightarrow f_{y_i} = \frac{\sin \phi}{\lambda} \approx \frac{y_i}{z_i \lambda} \rightarrow y_i = f_{y_i} \lambda z_i \]

In the region far from the aperture, the phase distribution is more like that of a planar wave.

Figure 2.2-4: A spherical wave may be approximated at points near the z axis and sufficiently far from the origin by a paraboloidal wave. For very far points, the spherical wave approaches the plane wave.

Zero spatial freq at \( y_i = 0 \)
Spatial frequency and propagation angle

When a plane wave of unity amplitude traveling in the $z$ direction is transmitted through a thin optical element with complex amplitude transmittance $f(x, y) = \exp[-j2\pi(\nu_x x + \nu_y y)]$ the wave is modulated by the harmonic function, so that $U(x, y, 0) = f(x, y)$. The incident wave is then converted into a plane wave with a wavevector at angles $\theta_x = \sin^{-1} \lambda\nu_x$ and $\theta_y = \sin^{-1} \lambda\nu_y$.

$\Lambda = \frac{1}{\nu_x}$

$\Lambda_x = 1/\nu_x$

$\theta_x = \sin^{-1} \lambda\nu_x$

$\alpha = \lambda\nu_x$

**Figure 4.1-2** A thin element whose amplitude transmittance is a harmonic function of spatial frequency $\nu_x$ (period $\Lambda_x = 1/\nu_x$) bends a plane wave of wavelength $\lambda$ by an angle $\theta_x = \sin^{-1} \lambda\nu_x = \sin^{-1}(\lambda/\Lambda_x)$. 
Fourier transform and Diffraction

The field at the observation point $P$ is comprised of contributions from an ensemble of fields radiated from all the point sources.

The field at $P$ from a point source with an infinitesimal area at $(x_0, y_0)$,

$$dE(x_i, y_i) = \frac{e^{jkr}}{r}E(x_0, y_0)dx_0\,dy_0$$

$$r = \sqrt{z_i^2 + (x_i - x_0)^2 + (y_i - y_0)^2}$$

The contribution of the spherical waves from all the point sources to $E(x_i, y_i)$ is

**Fresnel-Kirchhoff diffraction formula**

$$E(x_i, y_i) = K \iint \frac{e^{jkr}}{r}E(x_0, y_0)dx_0\,dy_0$$

$$K = \frac{1}{j\lambda}$$

**HUYGENS-FRESNEL CONSTRUCTION (Remind!!)**

The total contribution to the disturbance at $P$ is expressed as an area integral over the primary wavefront,

$$\psi(P) = A \frac{\exp \left(-i(\omega t - kr_0)\right)}{r_0} \iint \frac{\exp (iks)}{s} K(\chi)\,dS$$

$$\exp (iks)/s \Rightarrow \frac{1}{i\lambda} \{\exp(iks)/s\}$$

Spherical wave from source $P_0$

Huygens’ Secondary wavelets on the wavefront surface $S$

Obliquity factor: unity at $C$ where $\chi=0$, zero at high enough zone index
Diffraction under paraxial approx.

\[ r = \sqrt{z_i^2 + (x_i - x_0)^2 + (y_i - y_0)^2} \]

(paraxial) \[ z_i^2 \gg (x_i - x_0)^2 + (y_i - y_0)^2 \]

\[ r \approx z_i \left( 1 + \frac{(x_i - x_0)^2 + (y_i - y_0)^2}{2z_i^2} \right) \]

Fresnel region (near-field region)

\[ r = z_i + \frac{x_i^2 + y_i^2}{2z_i} \]

Fraunhofer region (far-field region)

\[ r = z_i + \frac{x_i^2 + y_i^2}{2z_i} \cdot \frac{x_i x_0 + y_i y_0}{z_i} \frac{x_0^2 + y_0^2}{2z_i} \]

Micro systems

<table>
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<tr>
<th>Full Wave Equations</th>
<th>Rayleigh-Sommerfeld &amp; Fresnel-Kirchhoff</th>
<th>Fresnel (near field)</th>
<th>Fraunhofer (far field)</th>
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<tr>
<td>z \gg \lambda</td>
<td>[ z \gg \frac{\pi}{4\lambda} \left[ (x - \xi)^2 + (y - \eta)^2 \right]^{\frac{1}{2}} ]</td>
<td>z \gg \frac{k(z^2 + \eta^2)}{2}</td>
<td>[ z \gg \frac{k(z^2 + \eta^2)}{2} ]</td>
</tr>
<tr>
<td>( (\xi, \eta) )</td>
<td>Vector Solutions</td>
<td>Scalar Approximations</td>
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</tr>
<tr>
<td>850 nm 1550 nm</td>
<td></td>
<td></td>
<td>- Assume planar wavefronts</td>
</tr>
<tr>
<td>966 ( \mu )m 791 ( \mu )m</td>
<td></td>
<td></td>
<td>- Assume parabolic wavefronts</td>
</tr>
<tr>
<td>4.8 mm 2.5 mm</td>
<td></td>
<td></td>
<td>Rayleigh-Sommerfeld Formulation - Assume Spherical wavefronts</td>
</tr>
</tbody>
</table>

Examples: 50 \( \mu \)m Aperture, 200 \( \mu \)m Observation, \( \lambda = 850 \) nm, \( \lambda = 1550 \) nm
Huygens-Fresnel principle

“Every unobstructed point of a wavefront, at a given instant in time, serves as a source of secondary wavelets (with the same frequency as that of the primary wave). The amplitude of the optical field at any point beyond is the superposition of all these wavelets (considering their amplitude and relative phase).”

Huygens’s principle:
By itself, it is unable to account for the details of the diffraction process. It is indeed independent of any wavelength consideration.

Fresnel’s addition of the concept of interference
After the Huygens-Fresnel principle …

**Fresnel’s shortcomings:**
He did not mention the existence of backward secondary wavelets, however, there also would be a reverse wave traveling back toward the source. He introduce a quantity of the obliquity factor, but he did little more than conjecture about this kind.

\[
\psi(P) = A \exp \left( -\frac{i(\omega t - kr_0)}{r_0} \right) \int \frac{\exp(iks)}{s} K(\chi) \, dS
\]

**Gustav Kirchhoff**: Fresnel-Kirchhoff diffraction theory
A more rigorous theory based directly on the solution of the differential wave equation. He, although a contemporary of Maxwell, employed the older elastic-solid theory of light. He found \( K(\chi) = (1 + \cos \theta)/2 \). \( K(0) = 1 \) in the forward direction, \( K(\pi) = 0 \) with the back wave.

\[
\psi(P) = -\left( \frac{ia}{2\lambda} \right) \int \left[ \frac{\exp(ikr)}{r} \right] \left[ \frac{\exp(iks)}{s} \right] [\cos (n,r) - \cos (n,s)] \, dS
\]

**Arnold Johannes Wilhelm Sommerfeld**: Rayleigh-Sommerfeld diffraction theory
A very rigorous solution of partial differential wave equation. The first solution utilizing the electromagnetic theory of light.

\[
\psi(P) = -\left( \frac{ia}{\lambda} \right) \int \left[ \frac{\exp(ikr)}{r} \right] \left[ \frac{\exp(iks)}{s} \right] \cos (n,s) \, dS
\]
Fraunhofer diffraction and Fourier transform

**Fraunhofer region** (far-field region)

\[ r = z_i + \frac{x_i^2 + y_i^2}{2z_i} - \frac{x_i x_o + y_i y_o}{z_i} \]

From the **Fresnel–Kirchhoff integral** of diffraction,

\[ E(x_i, y_i) = K \int \int \frac{e^{jkr}}{r} E(x_0, y_0) dx_0 dy_0 \]

\[ E(x_i, y_i) = \frac{1}{j \lambda z_i} e^{jkz_i \frac{(x_i^2 + y_i^2)}{2z_i}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E(x_0, y_0) e^{-j2\pi(f_x x_o + f_y y_o)} dx_0 dy_0 \]

with \( f_x = \frac{x_i}{\lambda z_i} \) and \( f_y = \frac{y_i}{\lambda z_i} \)

**2D Fourier transform** of the field in \((x,y)\) domain into \((f_x, f_y)\):

\[ F\{g(x,y)\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) e^{-j2\pi(f_x x + f_y y)} dx dy \]

The diffraction pattern becomes:

\[ E(x_i, y_i) = \frac{1}{j \lambda z_i} e^{jkz_i \frac{(x_i^2 + y_i^2)}{2z_i}} F\{E(x_0, y_0)\} \] \( f_x = \frac{x_i}{\lambda z_i}, f_y = \frac{y_i}{\lambda z_i} \)

**Fraunhofer diffraction pattern is the Fourier transform of the source field.**
Fresnel region (near-field region) \[ r \approx z_i \left( 1 + \frac{(x_i - x_0)^2 + (y_i - y_0)^2}{2z_i^2} \right) \]

\[
\exp\left[ \frac{jk}{2z} \left( (x_i - x_o)^2 + (y_i - y_o)^2 \right) \right] \\
= \exp\left[ \frac{jk}{2z} (x_i^2 + y_i^2) \right] \exp\left[ \frac{-jk}{z} (x_i x_o + y_i y_o) \right] \exp\left[ \frac{jk}{2z} (x_o^2 + y_o^2) \right]
\]

Hence, we may write

\[
E(x_i, y_i) = \frac{1}{j\lambda z_i} e^{\frac{jk}{2z_i} \left( x_i^2 + y_i^2 \right)} F \left\{ E(x_o, y_o) e^{\frac{jk}{2z_i} (x_o^2 + y_o^2)} \right\} \\
f_x = \frac{x_i}{\lambda z_i}, f_y = \frac{y_i}{\lambda z_i}
\]
Fourier optics

The branch of optics that can be analyzed by means of the Fourier transform

<table>
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<th>Signals in Time</th>
<th>Signals in Space</th>
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</thead>
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<tr>
<td>$g(t)$ signal</td>
<td>$g(x)$ signal</td>
</tr>
<tr>
<td>$t$ variable</td>
<td>$x$ variable</td>
</tr>
<tr>
<td>$f$ cyclic frequency</td>
<td>$f_x$ spatial frequency</td>
</tr>
<tr>
<td>$\omega = 2\pi f$ radian frequency</td>
<td>$k_x = 2\pi f_x$ wavenumber</td>
</tr>
</tbody>
</table>

Function | F-transform
---|---
DELTA FUNCTION | $\delta(\xi)$
1-D RECTANGULAR FUNCTION | $\delta(x)$
2-D RECTANGULAR FUNCTION | $\delta(x,\eta)$

Function | F-transform
---|---
ISOSCELES TRIANGLE | $\left(\frac{\sin x}{x}\right)^2$
GAUSSIAN | $e^{-\frac{1}{d^2}(x)^2}$
Fresnel diffraction and convolution

\[ g(x) \otimes f(x) = \int g(\xi) f(x-\xi) \, d\xi \]

Fresnel field can be expressed elegantly as a **CONVOLUTION** of two terms.

\[
E(x_i, y_i) = \frac{1}{j \lambda z_i} e^{j k [z_i + (x_i^2 + y_i^2) / 2z_i]} \left\{ E(x_0, y_0) \right\} e^{j k (x_0^2 + y_0^2) / 2z_i} \left\{ f_{x_0} = x_i / \lambda z_i, f_{y_0} = y_i / \lambda z_i \right\}
\]

\[
E(x_i, y_i) = \frac{j}{\lambda z_i} \iint E(x_0, y_0) \{ e^{j k [z_i + (x_i - x_0)^2 / 2z_i^2 + (y_i - y_0)^2 / 2z_i^2]} \} \, dx_0 \, dy_0
\]

The expression in the curly bracket is in the form of \( f(x_i - x_0, y_i - y_0) \)

From this observation, we note that the above expression takes on the form of a **convolution**:

\[ E(x_i, y_i) = E(x_i, y_i) \otimes f_{z_i}(x_i, y_i) \quad \text{where} \quad f_{z_i}(x_i, y_i) = \frac{1}{j \lambda z_i} e^{j k [z_i + (x_i^2 + y_i^2) / 2z_i]} \]

Point spread function (or impulse response of free space)

**PSF means** the field at \((x_i, y_i, z_i)\) when a point source is placed at the origin of the source coordinates.
Impulse response function of free space in Fresnel approximation

\[ f_{z_i}(x_i, y_i) = \frac{1}{j\lambda z_i} e^{jk[z_i + (x_i^2 + y_i^2)/2z_i]} \quad z_i = d, \text{ in general,} \quad h(x,y) \]

Therefore, free-space propagation can be treated as a convolution in the Fresnel approximation!

The impulse-response function \( h(x, y) \) of the system of free-space propagation is the response \( g(x, y) \) when the input \( f(x, y) \) is a point at the origin \((0, 0)\).

\[ h(x, y) \approx h_0 \exp \left[ -jk \frac{x^2 + y^2}{2d} \right], \quad \text{where} \quad h_0 = (j/\lambda d) \exp(-jkd). \]

\[ g(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x', y') h(x - x', y - y') \, dx' \, dy' \]

Therefore, free-space propagation can be treated as a convolution in the Fresnel approximation!

\[ g(x, y) = h_0 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x', y') \exp \left[ -j\pi \frac{(x - x')^2 + (y - y')^2}{\lambda d} \right] \, dx' \, dy' \]
**Impulse response function and transfer function**

The Fourier transform $F_{zi}$ of the point spread function $f_{zi}(x_i, y_i)$ is defined as:

$$f_{zi}(x_i, y_i) = \frac{1}{j\lambda z_i} e^{jkz_i + (x_i^2 + y_i^2)/2z_i}$$

**FT**

$$F_{zi} = e^{jkz_i - j\pi\lambda z_i (f_x^2 + f_y^2)}$$

**PSF (or, Impulse Response function)**

"Transfer function"

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**< proof >**

$$\mathcal{F}\{e^{-\pi x^2}\} = e^{-\pi f^2} \quad \text{The Fourier transform of this function is the original function itself.}$$

$$\mathcal{F}\{g(\alpha x)\} = \frac{1}{\alpha} G\left(\frac{f_x}{\alpha}\right) \quad \text{The similarity theorem of the Fourier transform}$$

$$e^{jkx_i^2/2z_i} = e^{[-\pi(x_i/\sqrt{j\lambda z_i})^2]}$$

$$\mathcal{F}\{e^{[-\pi(x_i/\sqrt{j\lambda z_i})^2]}\} = \sqrt{j\lambda z_i} e^{-j\pi\lambda z_i f_x^2}$$

$$\mathcal{F}\{e^{jk(x_i^2 + y_i^2)/2z_i}\} = j\lambda z_i e^{-j\pi\lambda z_i (f_x^2 + f_y^2)}$$

Hence, the Fourier transform of the point spread function is

$$\mathcal{F}\left\{\frac{1}{j\lambda z_i} e^{jkz_i + (x_i^2 + y_i^2)/2z_i}\right\} = e^{jkz_i - j\pi\lambda z_i (f_x^2 + f_y^2)}$$
Appendix: Transfer function

\[ F_2(\nu) = \mathcal{H}(\nu)F_1(\nu) \]

\( \mathcal{H}(\nu) \) is known as the system’s transfer function.

- **Ideal system**: \( \mathcal{H}(\nu) = 1 \) and \( h(t) = \delta(t) \); the output is a replica of the input.
- **Ideal system with delay**: \( \mathcal{H}(\nu) = \exp(-j2\pi\nu\tau) \) and \( h(t) = \delta(t - \tau) \); the output is a replica of the input delayed by time \( \tau \).
- **System with exponential response**: \( \mathcal{H}(\nu) = \tau/(1 + j2\pi\nu\tau) \) and \( h(t) = e^{-t/\tau} \) for \( t \geq 0 \), and \( h(t) = 0 \), otherwise; this represents the response of a system described by a first-order linear differential equation, e.g., that representing an \( R-C \) circuit with time constant \( \tau \). An impulse at the input results in an exponentially decaying response.
- **Chirped system**: \( \mathcal{H}(\nu) = \exp(-j\pi\nu^2) \) and \( h(t) = e^{-j\pi/4} \exp(j\pi t^2) \); the system distorts the input by imparting to it a phase shift proportional to \( \nu^2 \). An input impulse generates an output in the form of a chirped signal, i.e., a harmonic function whose instantaneous frequency (the derivative of the phase) increases linearly with time. This system describes the propagation of optical pulses through media with a frequency-dependent phase velocity (see Sec. 5.6). It also describes changes in the spatial distribution of light waves as they propagate through free space (see Sec. 4.1C).
Huygens’ wave front construction

Every point on a wave front is a source of secondary wavelets.
i.e. particles in a medium excited by electric field (E) re-radiate in all directions
i.e. in vacuum, E, B fields associated with wave act as sources of additional fields

Given wavefront at \( t \)

Allow wavelets to evolve for time \( \Delta t \)

New wavefront

Construct the wave front tangent to the wavelets

\[ r = c \Delta t \approx \frac{\lambda}{r} \]

What about \(-r\) direction?

(\(\pi\)-phase delay when the secondary wavelets, Hecht, 3.5.2, 3nd Ed)