20. Aberration Theory

- Wavefront aberrations (파면수차)
- Chromatic Aberration (색수차)
- Third-order (Seidel) aberration theory
  - Spherical aberrations
  - Coma
  - Astigmatism
  - Curvature of Field
  - Distortion
Five third-order (Seidel) aberrations

Chromatic

\[ n(\lambda) \]

Monochromatic

Unclear image

Deformation of image

Spherical

Coma

astigmatism

Distortion

Field Curvature
• Because the focal length of a lens depends on the refractive index (n), and this in turn depends on the wavelength, \( n = n(\lambda) \), light of different colors emanating from an object will come to a focus at different points.

• A white object will therefore not give rise to a white image. It will be distorted and have **rainbow edges**
Five monochromatic Aberrations

Unclear image
- Spherical
- Coma
- astigmatism

Deformation of image
- Distortion
- Field curvature
Spherical aberration

- This effect is related to rays which make large angles relative to the optical axis of the system.
- Mathematically, can be shown to arise from the fact that a lens has a spherical surface and not a parabolic one.
- Rays making significantly large angles with respect to the optic axis are brought to different foci.
Coma

- An **off-axis effect** which appears when a bundle of incident rays all make the same angle with respect to the optical axis (source at $\infty$)
- Rays are brought to a **focus at different points on the focal plane**
- Found in lenses with large spherical aberrations
- An off-axis object produces a **comet-shaped image**
Astigmatism and curvature of field

Yields elliptically distorted images
• This effect results from the difference in lateral magnification of the lens.

• If \( f \) differs for different parts of the lens,

\[
M_T = -\frac{s_i}{s_o} = \frac{y_i}{y_o}
\]

will differ also

- For \( f_i > 0 \) (positive lens), the magnification on axis is less than off axis:
  - M on axis less than off axis
  - Pincushion image

- For \( f_i < 0 \) (negative lens), the magnification on axis is greater than off axis:
  - M on axis greater than off axis
  - Barrel image
A mathematical treatment of the monochromatic aberrations can be developed by expanding the binomial series up to higher orders.

**Third-order aberration theory**

*Binomial power series:*

\[
(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + O(x^3)
\]

*Cosine power series:*

\[
\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + O(x^6)
\]

*Sine power series:*

\[
\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + O(x^7)
\]

*x* in the above expansions represents a number smaller than one, so that the succeeding terms involving \(x^2, x^3, \text{ and higher powers, get smaller and smaller.} \rightarrow \text{Paraxial approximation}\*

In our study of aberrations in this chapter, our basic expressions are power series written to the third order; thus, this study of aberrations is called third-order theory.
Now, let’s derive the expression of the third-order aberrations (Seidel aberrations, Ludwig von Seidel)
20-1. Ray and wave aberrations

Ideally, all rays that leave an object point and pass through an optical system should pass through the same image point.

Unfortunately, only when the rays are paraxial do they very nearly pass through the same image point; the paraxial image point.

As the rays deviate from their paraxial nature, they begin to miss the paraxial image point more and more.

LA: ray aberration - longitudinal
TA: ray aberration - transverse (lateral)
Longitudinal Ray Aberration

\[ n = 1.0 \quad n_L = 2.0 \]

\[ n_i \sin \theta_i = n_t \sin \theta_t \quad \Rightarrow \quad \sin \theta_t = \frac{\sin \theta_i}{2} \]
Longitudinal Aberration – cont’d

\[
\gamma'' = \pi - \theta_t - (\pi - \theta_i) = \theta_i - \theta_t
\]

\[
\sin \gamma'' = \sin (\theta_i - \theta_t) = \frac{R \sin \theta_t}{R + \Delta z}
\]

\[
n_i \sin \theta_i = n_t \sin \theta_t
\]

\[
\Delta Z = \Delta Z (\theta_i)
\]
Longitudinal Aberration – cont’d

\[ n = 1.0 \]

\[ n_L = 2.0 \]

\[
\sin(\theta_i - \theta_t) = \frac{R \sin \theta_t}{R + \Delta z}
\]

\[ n_i \sin \theta_i = n_t \sin \theta_t \]
Let’s start the aberration calculation for a simple case.

To the paraxial (first-order) ray approximation, $PQI = POI$ according to Fermat’s principle.

Beyond a first approximation, $PQI$ (depends on the position of $Q$) $\neq$ $POI$.

Thus we define the aberration at $Q$ as

$$a(Q) = (PQI - POI)_{opd} = (n_1\ell + n_2\ell') - (n_1s + n_2s')$$
Let's describe \( l \) in terms of \( R, s, \phi \).

\[
\ell^2 = \alpha^2 + \beta^2 \\
\cos \phi = \frac{\beta + R}{s + R} \quad \sin \phi = \frac{\alpha}{s + R} \\
\sin^2 \phi + \cos^2 \phi = 1 = \frac{(\beta + R)^2 + \alpha^2}{(s + R)^2} \quad \Rightarrow \quad (s + R)^2 = \beta^2 + 2\beta R + R^2 + \alpha^2 \\
(s + R)^2 = \ell^2 + 2\beta R + R^2 \quad \beta = (s + R) \cos \phi - R
\]

Substituting and rearranging we obtain:

\[
\ell^2 = (s + R)^2 + R^2 - 2R(s + R)\cos \phi
\]
Refraction at a spherical interface – cont’d

\[ \ell''^2 = (R + \alpha)^2 + \beta^2 \]

\[ \cos \phi = \frac{\alpha}{s' - R} \quad \sin \phi = \frac{\beta}{s' - R} \]

\[ \sin^2 \phi + \cos^2 \phi = 1 = \frac{\alpha^2 + \beta^2}{(s' - R)^2} \quad \Rightarrow \quad (s' - R)^2 = \beta^2 + \alpha^2 \]

\[ \ell''^2 = R^2 + 2\alpha R + \alpha^2 + \beta^2 = R^2 + 2\alpha R + (s' - R)^2 \]

\[ \alpha = (s' - R) \cos \phi \quad \Rightarrow \quad \ell''^2 = R^2 + (s' - R)^2 + 2R (s' - R) \cos \phi \]
Writing the $\cos \phi$ term in terms of $h$ we obtain:

$$
\cos \phi = \sqrt{1 - \sin^2 \phi} = \left[ 1 - \left( \frac{h}{R} \right)^2 \right]^{1/2}
\approx 1 - \frac{h^2}{2R^2} - \frac{h^4}{8R^4}
$$

where we have used the binomial expansion

$$(1 + x)^{1/2} \approx 1 + \frac{x}{2} - \frac{x^2}{8}$$

Substituting into our expressions for $\ell$ and $\ell'$ and rearranging

$$
\ell = s \left\{ 1 + \left[ \frac{h^2 (R + s)}{Rs^2} + \frac{h^4 (R + s)}{4R^3 s^2} \right] \right\}^{1/2}
$$

$$
\ell' = s' \left\{ 1 + \left[ \frac{h^2 (R - s')}{Rs'^2} + \frac{h^4 (R - s')}{4R^3 s'^2} \right] \right\}^{1/2}
$$

Use the same binomial expansion and neglecting terms of order $h^6$ and higher we obtain

$$
\ell = s \left\{ 1 + \frac{h^2 (R + s)}{2Rs^2} + \frac{h^4 (R + s)}{8R^3 s^2} - \frac{h^4 (R + s)^2}{8R^2 s^4} \right\}
$$

$$
\ell' = s' \left\{ 1 + \frac{h^2 (R - s')}{2Rs'^2} + \frac{h^4 (R - s')}{8R^3 s'^2} - \frac{h^4 (R - s')^2}{8R^2 s'^4} \right\}
$$

Figure 5-3 Refraction of a ray at a spherical surface.
Refraction at a spherical interface – cont’d

\[ \ell = s \left\{ \frac{1}{2} + \frac{h^2}{2s^2} + \frac{h^4}{8R^2 s^2} + \frac{h^4}{8R^3 s} - \frac{h^4}{8s^4} - \frac{h^4}{4Rs^3} - \frac{h^4}{8R^2 s^2} \right\} \]

\[ \ell' = s' \left\{ 1 + \frac{h^2}{2s'^2} - \frac{h^2}{2Rs'} + \frac{h^4}{8R^2 s'^2} - \frac{h^4}{8R^3 s'} - \frac{h^4}{8s'^4} + \frac{h^4}{4Rs'^3} - \frac{h^4}{8R^2 s'^2} \right\} \]

\[ a(Q) = (n_1 \ell + n_2 \ell') - (n_1 s + n_2 s') \]

Imaging formula (first-order approx.)
\[ \frac{n_1}{s} + \frac{n_2}{s'} = \frac{n_2 - n_1}{R} \]

\[ a(Q) = -\frac{h^4}{8} \left[ \frac{n_1}{s} \left( \frac{1}{s} + \frac{1}{R} \right)^2 + \frac{n_2}{s'} \left( \frac{1}{s'} - \frac{1}{R} \right)^2 \right] = ch^4 \]

Aberration for axial object points (on-axis imaging)

: This aberration will be referred to as *spherical aberration*.

→ The other aberrations will appear at off-axis imaging!
20-3. Spherical Aberration

\[ a \left( Q \right) = c h^4 \]

\[ b_y = \frac{4 c s'}{n_2} h^3 \quad b_z = \frac{4 c s'^2}{n_2} h^2 \]

Optics, E. Hecht, p. 222.
Spherical Aberration

Modern Optics, R. Guenther, p. 196.
Spherical Aberration

For a thin lens with surfaces with radii of curvature $R_1$ and $R_2$, refractive index $n_L$, object distance $s$, image distance $s'$, the difference between the paraxial image distance $s'_p$ and image distance $s'_h$ is given by

$$\frac{1}{s'_h} - \frac{1}{s'_p} = \frac{h^2}{8 f^3 n_L (n_L - 1)} \left[ \frac{n_L + 2}{n_L - 1} \sigma^2 + 4 (n_L + 1) p \sigma + (3n_L + 2)(n_L - 1) p^2 + \frac{n_L^3}{n_L - 1} \right]$$

where,

$$\sigma = \frac{R_2 + R_1}{R_2 - R_1} \text{(shape factor)}, \quad p = \frac{s' - s}{s' + s}$$

\textbf{Spherical aberration is minimized when}:

$$\sigma = - \frac{2(n_L^2 - 1)}{n_L + 2} p$$
Spherical aberration is minimized when:

\[
\sigma = -\frac{2\left(n_L^2 - 1\right)}{n_L + 2} p
\]

\[
\sigma = \frac{R_2 + R_1}{R_2 - R_1}, \quad p = \frac{s' - s}{s' + s}
\]

For an object at infinite (\( p = -1, n_L = 1.50 \)), \( \sigma \sim 0.7 \).
Spherical Aberration

A graph of the spherical aberration for lenses of different shape but the same focal length. For the lenses shown $h = 1 \text{ cm, } f = +10 \text{ cm, } d = 2 \text{ cm, and } n = 1.51700.$
Third-Order Aberration: Off-axis imaging by a spherical interface

Now, let’s calculate the third-order aberrations in a general case.

\[ a'(Q) = (PQP' - PBP')_{opd} = c(BQ)^4 = c\rho'^4 \]
\[ a'(O) = (POP' - PBP')_{opd} = c(BO)^4 = cb^4 \]

\[ a(Q) = a'(Q) - a'(O) = c(\rho'^4 - b^4) \]
\[ \rho'^2 = r^2 + b^2 + 2rb \cos \theta, \quad b \propto h' \]
\[ \Rightarrow a(Q) = a(Q; r, \theta, h') \]
After some very complicated analysis the third-order aberration equation is obtained:

$$a(Q) = C_{40} r^4 + C_{31} h' r^3 \cos \theta + C_{22} h'^2 r^2 \cos^2 \theta + C_{20} h'^2 r^2 + C_{11} h'^3 r \cos \theta$$

- Spherical Aberration
- Coma
- Astigmatism
- Curvature of Field
- Distortion

On-axis imaging 에서의 $a(Q) = ch^4$ 와 일치
20-4. Coma

\[ a(Q) = 1 C_{31} h' r^3 \cos \theta \quad (h' \neq 0, \ \cos \theta \neq 0) \]
Coma

Figure 6.18  Positive coma. (Photo by E.H.)
Coma

Least Coma

Most Coma

Modern Optics, R. Guenther, p. 205.
Coma
20-5. Astigmatism

\[ a(Q) = 2C_{22} h'^2 r^2 \cos^2 \theta \quad (h' \neq 0, \cos \theta \neq 0) \]

Optics, E. Hecht, p. 224.
Astigmatism

Sagittal plane

Tangential plane (Meridional plane)

tangential line image plane
circle image plane
sagittal line image plane
paraxial image plane
Astigmatism

Coma
Astigmatism

Least Astig.

Most Astig.

Modern Optics, R. Guenther, p. 207.
Field Curvature

\[ a(Q) = 2C_{20} h'^2 r^2 \quad (h' \neq 0) \]

Astigmatism when \( \theta = 0 \).

A flat object normal to the optical axis cannot be brought into focus on a flat image plane.

This is less of a problem when the imaging surface is spherical, as in the human eye.
Astigmatism when $\theta = 0$.

If no astigmatism is present, the sagittal and tangential image surfaces coincide on the Petzval surface.

The best image plane, Petzval surface, is actually not planar, but spherical.

This aberration is called field curvature.
20-6. Distortion

\[ a(Q) = 3C_{11} h^3 r \cos \theta \quad (h' \neq 0, \ \cos \theta \neq 0) \]

**Figure 5-10** Images of a square grid (a) showing pincushion distortion (b) and barrel distortion (c) due to nonuniform magnifications.
20-7. Chromatic Aberration

Figure 6.32  Axial chromatic aberration.

Optics, E. Hecht, p. 232.
Achromatic Doublet

Figure 20-13  Achromatic doublet, consisting of (1) crown glass equiconvex lens cemented to (2) a negative flint glass lens. Notation for the four radii of curvature are shown.

Figure 20-13  Power of two lenses, $P_{1D}$ and $P_{2D}$ are different at the Fraunhofer wavelength, $\lambda_D = 587.6$ nm.

* Note: What is the definition of the Fraunhofer wavelengths (lines)?
* Note: Fraunhofer wavelengths (lines)

Spectrum of a blue sky somewhat close to the horizon pointing east at around 3 or 4 pm on a clear day.

The dark lines in the solar spectrum were caused by absorption by those elements in the upper layers of the Sun.

<table>
<thead>
<tr>
<th>Designation</th>
<th>Element</th>
<th>Wavelength (nm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>Hα</td>
<td>656.281</td>
</tr>
<tr>
<td>F</td>
<td>Hβ</td>
<td>486.134</td>
</tr>
<tr>
<td>D₁</td>
<td>Na</td>
<td>589.592</td>
</tr>
<tr>
<td>D₂</td>
<td>Na</td>
<td>589.995</td>
</tr>
<tr>
<td>D₃ or d</td>
<td>He</td>
<td>587.5618</td>
</tr>
</tbody>
</table>
Achromatic Doublets

Chromatic aberration is eliminated when:

\[
\frac{\partial P}{\partial \lambda} = K_1 \frac{\partial n_1}{\partial \lambda} + K_2 \frac{\partial n_2}{\partial \lambda} = 0
\]

In general, for materials with "normal" dispersion:

\[
\frac{\partial n}{\partial \lambda} < 0
\]

That means that to eliminate chromatic aberration, \(K_1\) and \(K_2\) must have opposite signs.

The partial derivative of refractive index with wavelength is approximated:

\[
\frac{\partial n}{\partial \lambda} \approx \frac{n_F - n_C}{\lambda_F - \lambda_C}
\]

for an achromat in the visible region of the spectrum.

In the above equation, \(\lambda_F = 486.1\) nm and \(\lambda_C = 656.3\) nm.

For a lens separation of \(L\): Total power is

\[
P = \frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{L}{f_1 f_2} \Rightarrow P = P_1 + P_2 - L P_1 P_2
\]

For a cemented doublet \(L = 0\):

\[
P = P_1 + P_2 = (n_1 - 1) K_1 + (n_2 - 1) K_2
\]

\(\lambda_D = 587.6\) nm = center of visible spectrum
Achromatic Doublets

Defining the dispersive constant $V$:

$$V = \frac{n_D - 1}{n_F - n_C}$$

we can write

$$K_1 \frac{\partial n_{1D}}{\partial \lambda} = K_1 \left( \frac{n_{1F} - n_{1C}}{\lambda_f - \lambda_c} \right) \left( \frac{n_{1D} - 1}{n_{1D} - 1} \right) = \frac{P_{1D}}{(\lambda_f - \lambda_c)V_1}$$

$$K_2 \frac{\partial n_{2D}}{\partial \lambda} = K_2 \left( \frac{n_{2F} - n_{2C}}{\lambda_f - \lambda_c} \right) \left( \frac{n_{2D} - 1}{n_{2D} - 1} \right) = \frac{P_{2D}}{(\lambda_f - \lambda_c)V_2}$$

$V_1$ and $V_2$ are functions only of the material properties of the two lenses.

$$\frac{P_{1D}}{(\lambda_f - \lambda_c)V_1} + \frac{P_{2D}}{(\lambda_f - \lambda_c)V_2} = 0 \quad \Rightarrow \quad V_2 P_{1D} + V_1 P_{2D} = 0$$

Achromatic condition for doublets
We can solve for the power of each lens in terms of the desired power of the doublet:

\[ P_D = P_{1D} + P_{2D} \]

\[ V_2 P_{1D} + V_1 P_{2D} = 0 \quad \Rightarrow \quad P_{1D} = -P_D \frac{V_1}{V_2 - V_1} \quad P_{2D} = P_D \frac{V_2}{V_2 - V_1} \]

where \( P_D = (n_{1D} - 1) K_1 + (n_{2D} - 1) K_2 \)

\[ K_1 = \frac{P_{1D}}{(n_{1D} - 1)} \quad , \quad K_2 = \frac{P_{2D}}{(n_{2D} - 1)} \]

From the values of \( K_1 \) and \( K_2 \), the 4 radii of curvature for the two surfaces of the lenses can be determined.

If lens 1 is bi-convex with equal curvature for each surface:

\[ r_{12} = -r_{11} \quad , \quad r_{21} = r_{12} \quad , \quad r_{22} = \frac{r_{12}}{1 - K_2 r_{12}} \quad \text{where,} \quad K_2 = \left( \frac{1}{r_{21}} - \frac{1}{r_{22}} \right) \]
**Achromatic Doublets**

**Figure 5-13** Achromatic doublet, consisting of (1) crown glass equiconvex lens cemented to (2) a negative flint glass lens. Notation for the four radii of curvature are shown.

<table>
<thead>
<tr>
<th>Table 20-1 SAMPLE OF OPTICAL GLASSES</th>
</tr>
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<tbody>
<tr>
<td>Type</td>
</tr>
<tr>
<td>---------------------</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td>Borosilicate crown</td>
</tr>
<tr>
<td>Borosilicate crown</td>
</tr>
<tr>
<td>Light barium crown</td>
</tr>
<tr>
<td>Dense barium crown</td>
</tr>
<tr>
<td>Dense flint</td>
</tr>
<tr>
<td>Flint</td>
</tr>
<tr>
<td>Dense flint</td>
</tr>
<tr>
<td>Dense flint</td>
</tr>
<tr>
<td>Fused silica</td>
</tr>
</tbody>
</table>