20. Fresnel equations

- EM Waves at an Interface
- Fresnel Equations: Reflection and Transmission Coefficients
- Brewster’s Angle
- Total Internal Reflection
- Evanescent Waves
- The Complex Refractive Index
- Reflection from Metals
We will derive the Fresnel equations

$r : \text{reflection coefficient}$

$\text{TE} : \quad r = \frac{E_r}{E} = \frac{\cos \theta - \sqrt{n^2 - \sin^2 \theta}}{\cos \theta + \sqrt{n^2 - \sin^2 \theta}}$

$\text{TM} : \quad r = \frac{E_r}{E} = \frac{n^2 \cos \theta - \sqrt{n^2 - \sin^2 \theta}}{n^2 \cos \theta + \sqrt{n^2 - \sin^2 \theta}}$

$t : \text{transmission coefficient}$

$\text{TE} : \quad t = \frac{E_t}{E} = \frac{2 \cos \theta}{\cos \theta + \sqrt{n^2 - \sin^2 \theta}}$

$\text{TM} : \quad t = \frac{E_t}{E} = \frac{2n \cos \theta}{n^2 \cos \theta + \sqrt{n^2 - \sin^2 \theta}}$
Figure 20-1  Defining diagram for incident, reflected, and transmitted rays at an $XY$-plane interface when the electric field is perpendicular to the plane of incidence, the TE mode.
EM Waves at an Interface

Incident beam: \[ \vec{E}_i = \vec{E}_{oi} \exp\left[i\left(\vec{k}_i \cdot \vec{r} - \omega t\right)\right] \]

Reflected beam: \[ \vec{E}_r = \vec{E}_{or} \exp\left[i\left(\vec{k}_r \cdot \vec{r} - \omega t\right)\right] \]

Transmitted beam: \[ \vec{E}_t = \vec{E}_{ot} \exp\left[i\left(\vec{k}_t \cdot \vec{r} - \omega t\right)\right] \]

At the boundary between the two media (the \(x-y\) plane), all waves must exist simultaneously, and the tangential component must be equal on both sides of the interface.

Therefore, for all \(t\) and for all \(\vec{r}\) on the interface,

\[ \hat{n} \times \vec{E}_i + \hat{n} \times \vec{E}_r = \hat{n} \times \vec{E}_t \]

\[ \hat{n} \times \vec{E}_{oi} \exp\left[i\left(\vec{k}_i \cdot \vec{r} - \omega t\right)\right] + \hat{n} \times \vec{E}_{or} \exp\left[i\left(\vec{k}_r \cdot \vec{r} - \omega t\right)\right] = \hat{n} \times \vec{E}_{ot} \exp\left[i\left(\vec{k}_t \cdot \vec{r} - \omega t\right)\right] \]

Assuming that the wave amplitudes are constant, the only way that this can be true over the entire interface and for all \(t\) is if:

\[ \left(\vec{k}_i \cdot \vec{r} - \omega t\right) = \left(\vec{k}_r \cdot \vec{r} - \omega t\right) = \left(\vec{k}_t \cdot \vec{r} - \omega t\right) \] : Phase matching at the boundary!
EM Waves at an Interface

\[(\vec{k}_i \cdot \vec{r} - \omega_i t) = (\vec{k}_r \cdot \vec{r} - \omega_r t) = (\vec{k}_i \cdot \vec{r} - \omega_i t)\]

At \( \vec{r} = 0 \), this results in
\[\omega_i t = \omega_r t = \omega_i t\]
\[\Rightarrow \omega_i = \omega_r = \omega_i \] (Frequency does not change at the boundary!)

At \( t = 0 \), this results in
\[\Rightarrow \vec{k}_i \cdot \vec{r} = \vec{k}_r \cdot \vec{r} = \vec{k}_i \cdot \vec{r} \] (Phases on the boundary does not change!)

Subtracting any pair of these factors results in
\[\left(\vec{k}_i - \vec{k}_r\right) \cdot \vec{r} = \left(\vec{k}_i - \vec{k}_i\right) \cdot \vec{r} = \left(\vec{k}_l - \vec{k}_r\right) \cdot \vec{r} = 0\]

This equation \( \vec{k}_i \cdot \vec{r} = \text{constant} \) is the equation for a plane perpendicular to \( \vec{k}_i \cdot \vec{r} \). Consequently the above relation implies that \( \vec{k}_i, \vec{k}_r, \) and \( \vec{k}_l \) are coplanar in the plane of incidence.
At \( t = 0 \),
\[
\vec{k}_i \cdot \vec{r} = \vec{k}_r \cdot \vec{r} = \vec{k}_t \cdot \vec{r} = \text{constant}
\]

Considering the relation for the incident and reflected beams,
\[
\vec{k}_i \cdot \vec{r} = \vec{k}_r \cdot \vec{r} \quad \Rightarrow \quad k_i r \sin \theta_i = k_r r \sin \theta_r
\]

Since the incident and reflected beams are in the same medium,

\[
k_i = k_r = \frac{n_i \omega}{c} \quad \Rightarrow \quad \sin \theta_i = \sin \theta_r \quad \Rightarrow \quad \theta_i = \theta_r : \text{law of reflection}
\]

Considering the relation for the incident and transmitted beams,
\[
\vec{k}_i \cdot \vec{r} = \vec{k}_t \cdot \vec{r} \quad \Rightarrow \quad k_i r \sin \theta_i = k_t r \sin \theta_t
\]

But the incident and transmitted beams are in different media,

\[
k_i = \frac{n_i \omega}{c} \quad k_t = \frac{n_t \omega}{c} \quad \Rightarrow \quad n_i \sin \theta_i = n_t \sin \theta_t : \text{law of refraction}
\]
From Maxwell's EM field theory, we have the boundary conditions at the interface for the TE case:

\[ E_i + E_r = E_i \]
\[ B_i \cos \theta_i - B_r \cos \theta_r = B_i \cos \theta_i \]

The above conditions imply that the tangential components of both \( \vec{E} \) and \( \vec{B} \) are equal on both sides of the interface. We have also assumed that \( \mu_i \approx \mu_r \approx \mu_0 \), as is true for most dielectric materials.

For the TM mode:

\[ -E_i \cos \theta_i + E_r \cos \theta_r = -E_i \cos \theta_i \]
\[ B_i + B_r = B_i \]
Recall that \( E = \nu B = \left(\frac{c}{n}\right)B \Rightarrow B = \frac{nE}{c} \)

Let \( n_1 = \) refractive index of incident medium
\( n_2 = \) refractive index of refracting medium

For the TE mode:

\[
E_i + E_r = E_t \\
n_1E_i \cos \theta_i - n_1E_r \cos \theta_r = n_2E_t \cos \theta_i
\]

For the TM mode:

\[
-E_i \cos \theta_i + E_r \cos \theta_r = -E_i \cos \theta_i \\
n_1E_i + n_1E_r = n_2E_t
\]
Eliminating $E_\ell$ from each set of equations and solving for the reflection coefficient we obtain

**TE case:** \[ r = \frac{E_r}{E_i} = \frac{\cos \theta_i - n \cos \theta_i}{\cos \theta_i + n \cos \theta_i} \]

**TM case:** \[ r = \frac{E_r}{E_i} = \frac{n \cos \theta_i - \cos \theta_i}{n \cos \theta_i + \cos \theta_i} \]

where \[ n = \frac{n_2}{n_1} \]

We know that

\[ \sin \theta_i = n \sin \theta_i \]

\[ n \cos \theta_i = n \sqrt{1 - \sin^2 \theta_i} = n \sqrt{\frac{\sin^2 \theta_i}{n^2}} = \sqrt{n^2 - \sin^2 \theta_i} \]
Now we have derived the Fresnel Equations

Substituting we obtain the Fresnel equations for reflection coefficients $r$:

**TE case**: $r = \frac{E_r}{E_i} = \frac{\cos \theta_i - \sqrt{n^2 - \sin^2 \theta_i}}{\cos \theta_i + \sqrt{n^2 - \sin^2 \theta_i}}$

**TM case**: $r = \frac{E_r}{E_i} = \frac{n^2 \cos \theta_i - \sqrt{n^2 - \sin^2 \theta_i}}{n^2 \cos \theta_i + \sqrt{n^2 - \sin^2 \theta_i}}$

For the transmission coefficient $t$:

**TE case**: $t = \frac{E_t}{E_i} = \frac{2 \cos \theta_i}{\cos \theta_i + \sqrt{n^2 - \sin^2 \theta_i}}$

**TM case**: $t = \frac{E_t}{E_i} = \frac{2n \cos \theta_i}{n^2 \cos \theta_i + \sqrt{n^2 - \sin^2 \theta_i}}$

**TE**: $t = r + 1$

**TM**: $nt = r + 1$

These mean just the boundary conditions
Power : Reflectance(R) and Transmittance(T)

The quantities \( r \) and \( t \) are ratios of electric field amplitudes.

The ratios \( R \) and \( T \) are the ratios of reflected and transmitted powers, respectively, to the incident power:

\[
R = \frac{P_r}{P_i}, \quad T = \frac{P_t}{P_i}
\]

From conservation of energy:

\[
P_i = P_r + P_t \quad \Rightarrow \quad 1 = R + T
\]

We can express the power in each of the fields in terms of the product of an irradiance and area:

\[
P_i = I_i A_i, \quad P_r = I_r A_r, \quad P_t = I_t A_t
\]

\[
\Rightarrow \quad I_i A_i = I_r A_r + I_t A_t
\]

\[
I_i \cos \theta_i = I_r \cos \theta_r + I_t \cos \theta_t
\]

\[
I_i \cos \theta_i = I_r \cos \theta_r + I_t \cos \theta_t
\]

But \( I = \frac{1}{2} n \varepsilon_0 c E_0^2 \) \( \Rightarrow \)

\[
\frac{1}{2} n \varepsilon_0 c E_0^2 \cos \theta_i = \frac{1}{2} n \varepsilon_0 c E_0^2 \cos \theta_r + \frac{1}{2} n \varepsilon_0 c E_0^2 \cos \theta_t
\]

\[
\Rightarrow \quad I = E_{o_r}^2 + \frac{n_2 E_0^2 \cos \theta_r}{r \cos \theta_i} = \frac{E_{o_t}^2}{r \cos \theta_i} \Rightarrow R + T
\]

\[
R = \frac{E_{o_r}^2}{E_{o_i}^2} = r^2 \quad T = n \left( \frac{\cos \theta_t}{\cos \theta_i} \right) \frac{E_{o_t}^2}{E_{o_i}^2} = n \left( \frac{\cos \theta_t}{\cos \theta_i} \right) t^2
\]

\[
R = r r^* = |r|^2 \quad T = \left( n \frac{\cos \theta_t}{\cos \theta_i} \right) t t^* = \left( n \frac{\cos \theta_t}{\cos \theta_i} \right) |t|^2
\]
20-2. External and Internal Reflection

External reflection: \( n_1 < n_2 \implies n = \frac{n_2}{n_1} > 1 \)

Internal reflection: \( n_1 > n_2 \implies n = \frac{n_2}{n_1} < 1 \)

For internal reflection,
\[ \sqrt{n^2 - \sin^2 \theta_i} \] may be an imaginary number \( \implies \) total internal reflection.

Note Brewster's angle \( \theta_p \) (for polarizing angle) for the TM case:
\[ \implies r(\theta_p) = 0 \text{ when } \theta_p = \tan^{-1} n \]

\( \implies \) No reflection of TM mode for Brewster's angle.
Derivation of Brewster’s Angle

Brewster's angle $\theta_p$ (for polarizing angle) for the TM case:

$$r(\theta_p) = 0 = \frac{n^2 \cos \theta_p - \sqrt{n^2 - \sin^2 \theta_p}}{n^2 \cos \theta_p + \sqrt{n^2 - \sin^2 \theta_p}}$$

$$\Rightarrow \quad n^4 \cos^2 \theta_p = n^2 - \sin^2 \theta_p$$

$$n^4 \cos^2 \theta_p - n^2 + \sin^2 \theta_p = 0$$

$$\Rightarrow \quad n^2 = \frac{1 \pm \sqrt{1 - 4 \cos^2 \theta_p \sin^2 \theta_p}}{2 \cos^2 \theta_p} = \frac{1 \pm \sqrt{1 - 4(1 - \sin^2 \theta_p) \sin^2 \theta_p}}{2 \cos^2 \theta_p}$$

$$= \frac{1 \pm \sqrt{1 - 4 \sin^2 \theta_p + 4 \sin^4 \theta_p}}{2 \cos^2 \theta_p} = \frac{1 \pm (1 - 2 \sin^2 \theta_p)}{2 \cos^2 \theta_p}$$

$$n^2 = \frac{\sin^2 \theta_p}{\cos^2 \theta_p} \quad \Rightarrow \quad n = \frac{\sin \theta_p}{\cos \theta_p} \quad \Rightarrow \quad \theta_p = \tan^{-1} n$$

*For $n = 1.50$, $\theta_p = 56.31^\circ$*
For internal reflection: \( n = \frac{n_2}{n_1} < 1 \)

For \( \theta = \theta_c = \sin^{-1} n \), \( r = 1 \) for both (TE and TM) cases.

For \( \theta > \theta_c \), called total internal reflection (TIR),

\[ \Rightarrow r \text{ is a complex number} \]

\[ \Rightarrow R = rr^* = 1. \]

**TE case**: \[ r = \frac{E_r}{E_i} = \frac{\cos \theta_i - i\sqrt{\sin^2 \theta_i - n^2}}{\cos \theta_i + i\sqrt{\sin^2 \theta_i - n^2}} \]

**TM case**: \[ r = \frac{E_r}{E_i} = \frac{n^2 \cos \theta_i - i\sqrt{n^2 \sin^2 \theta_i - n^2}}{n^2 \cos \theta_i + i\sqrt{n^2 \sin^2 \theta_i - n^2}} \]
20-3. Phase changes on External Reflection

When \( r \) is a real number, as it always is for external reflection, then the phase shift is \( 0^\circ \) for \( r > 0 \), and the phase shift is \( 180^\circ (= \pi) \) for \( r < 0 \).

For \( r < 0 \):

\[
E_r = -|r|E_i
\]

\[
= \exp(i\pi)|r|E_0\exp\left[i(\vec{k} \cdot \vec{r} - \omega t)\right]
\]

\[
= |r|E_0\exp\left[i(\vec{k} \cdot \vec{r} - \omega t + \pi)\right]
\]

수직입사인 경우

\( \pi \)-phase 변화, 즉, 위상 반전이 일어나는가?
Why two $r$’s are opposite in their signs at $\theta_i \to 0$?

- $r_{TM} > 0 \quad \Rightarrow \quad E_3$ is opposite to $E_1$
- $r_{TE} < 0 \quad \Rightarrow \quad E_3$ is opposite to $E_1$

The opposite signs in $r$’s correspond completely to the phase shift of $\pi$ after reflection in both cases.
Phase Shifts for Internal Reflection

When \( \theta < \theta_c \) then \( r \) is a real number and the phase shift will be 0° for \( r > 0 \) and 180° for \( r < 0 \).

When \( \theta \geq \theta_c \) (TIR case) then \( r \) is complex and for both the TE and TM cases has the form:

\[
r = \frac{a - ib}{a + ib} = \frac{\cos \alpha - i \sin \alpha}{\cos \alpha + i \sin \alpha} = e^{-ia} = e^{-i2a} = e^{-i\phi}
\]

\[
\Rightarrow \tan \alpha = \frac{\sin \alpha}{\cos \alpha} = \frac{b}{a} \quad \phi = 2\alpha
\]

\( \phi \) is the phase shift for total internal reflection (TIR).

**TE case:**
\[
r = \frac{E_r}{E_i} = \frac{\cos \theta_t - i\sqrt{\sin^2 \theta_t - n^2}}{\cos \theta_t + i\sqrt{\sin^2 \theta_t - n^2}}
\]

\[
a = \cos \theta_t \quad b = \sqrt{\sin^2 \theta_t - n^2}
\]

\[
\Rightarrow \tan \alpha = \tan\left(\frac{\phi}{2}\right) = \frac{\sqrt{\sin^2 \theta_t - n^2}}{\cos \theta}
\]

\[
\phi_{TE} = 2\tan^{-1}\left(\frac{\sqrt{\sin^2 \theta_t - n^2}}{\cos \theta}\right)
\]

A similar analysis for the TM case gives:

\[
\phi_{TM} = 2\tan^{-1}\left(\frac{\sqrt{\sin^2 \theta_t - n^2}}{n^2 \cos \theta}\right)
\]
Phase Shifts for Internal Reflection

\[ \phi_{TM} = 2 \tan^{-1} \left( \frac{\sqrt{\sin^2 \theta_i - n^2}}{n^2 \cos \theta} \right) \]

\[ \phi_{TE} = 2 \tan^{-1} \left( \frac{\sqrt{\sin^2 \theta_i - n^2}}{\cos \theta} \right) \]
Phase Shifts for Internal Reflection

\[ \phi_{TM} = \begin{cases} 
180^\circ (\pi) & \theta < \theta_p \\
0^\circ & \theta_p < \theta < \theta_c \\
2\tan^{-1} \left( \frac{\sin^2 \theta_i - n^2}{n^2 \cos \theta} \right) & \theta < \theta_c 
\end{cases} \]

\[ \phi_{TE} = \begin{cases} 
0^\circ & \theta < \theta_c \\
2\tan^{-1} \left( \frac{\sin^2 \theta_i - n^2}{\cos \theta} \right) & \theta > \theta_c 
\end{cases} \]

\[ \phi_{TM} - \phi_{TE} : \begin{cases} 
= 0^\circ & \theta < \theta_c \\
> 0^\circ & \theta > \theta_c 
\end{cases} \]

Note \( \phi_{TM} - \phi_{TE} = 45^\circ \) near \( \theta_i = 53^\circ \) when \( n = 1.5 \)
Note $\phi_{TM} - \phi_{TE} = 45^\circ$ at $\theta_i = 53^\circ$ when $n = 1.5$
20-5. Evanescent Waves at an Interface

Incident beam: \( \vec{E}_i = \vec{E}_{oi} \exp\left[i\left(\vec{k}_i \cdot \vec{r} - \omega t\right)\right] \)

Reflected beam: \( \vec{E}_r = \vec{E}_{or} \exp\left[i\left(\vec{k}_r \cdot \vec{r} - \omega t\right)\right] \)

Transmitted beam: \( \vec{E}_t = \vec{E}_{ot} \exp\left[i\left(\vec{k}_t \cdot \vec{r} - \omega t\right)\right] \)

For the transmitted beam:
\[
E_t = E_{ot} \exp\left[i\left(\vec{k}_t \cdot \vec{r} - \omega t\right)\right]
\]

\[
\vec{k}_t \cdot \vec{r} = (k_i \sin \theta + k_i \cos \theta \cdot \bar{y}) \cdot (x \bar{x} + y \bar{y})
= k_i (x \sin \theta + y \cos \theta)
\]

But, \( \cos \theta_t = \sqrt{1 - \sin^2 \theta_t} = \sqrt{1 - \frac{\sin^2 \theta}{n}} \)

When \( \sin \theta_i > n \) \( \text{ (total internal reflection), then}: \)
\[
\cos \theta_i = i \sqrt{\frac{\sin^2 \theta_i}{n}} - 1 \Rightarrow \text{ a purely imaginary number} 
\]
**Evanescent Waves at an Interface**

For the transmitted beam with an TIR condition \((\sin \theta_1 > n)\), we can write the phase factor as:

\[
\vec{k}_t \cdot \vec{r} = k_t \left( x \frac{\sin \theta_1}{n} + i y \sqrt{\frac{\sin^2 \theta_1}{n} - 1} \right)
\]

Defining the coefficient \(\alpha\):

\[
\alpha = k_t \sqrt{\frac{\sin^2 \theta_1}{n} - 1} = \frac{2\pi}{\lambda_t} \sqrt{\frac{\sin^2 \theta_1}{n} - 1}
\]

We can write the transmitted wave as:

\[
E_t = E_{0t} \exp \left[ i \left( \frac{k_t x \sin \theta_1}{n} - \omega t \right) \right] \exp(-\alpha y)
\]

The evanescent wave amplitude will decay rapidly as it penetrates into the lower refractive index medium.

**Penetration depth:**

\[
E_i = \left( \frac{1}{e} \right) E_{0t} \Rightarrow h = \frac{1}{\alpha} = \frac{\lambda}{2\pi} \sqrt{\frac{\sin^2 \theta_1}{n^2} - 1}
\]

Note that the incident and reflection waves form a standing wave in x direction.
Frustrated TIR

\( T_p = \) fraction of intensity transmitted across gap

\[ T_p = \frac{1}{(\alpha \sinh^2 \gamma + 1)} \]

\[ \alpha = \left( \frac{n^2 - 1}{2n} \right)^2 \left[ (n^2 + 1) \sin^2 \theta_i - 1 \right] \left( \frac{\cos^2 \theta_i (n^2 \sin^2 \theta_i - 1)}{n^2 \sin^2 \theta_i - 1} \right) \]

\[ \gamma = 2\pi \left( \frac{d}{\lambda} \right) (n^2 \sin^2 \theta_i - 1)^{1/2} \]


Fig. 2. (a) Tunneling of light through the gap between the regions 1 and 2; frustrated total internal reflection. (b) The fraction of transmitted light vs \((d/\lambda)\) plotted for two different values of the refractive index \(n\).
For a material with conductivity ($\sigma$)

$$\tilde{n} = \sqrt{1 + i \left( \frac{\sigma}{\varepsilon_0 \omega} \right)} = n_R + i n_I$$

$$\tilde{n}^2 = 1 + i \left( \frac{\sigma}{\varepsilon_0 \omega} \right) = n_R^2 - n_I^2 + i 2n_R n_I$$

Solving for the real and imaginary components we obtain:

$$n_R^2 - n_I^2 = 1 \quad 2n_R n_I = \frac{\sigma}{\varepsilon_0 \omega} \quad \Rightarrow \quad n_R = \frac{\sigma}{2 n_I \varepsilon_0 \omega}$$

$$\Rightarrow \quad \left( \frac{\sigma}{2 n_I \varepsilon_0 \omega} \right)^2 - n_I^2 = 1 \quad \Rightarrow \quad n_I^4 - n_I^2 - \left( \frac{\sigma}{2 \varepsilon_0 \omega} \right)^2 = 0$$

From the quadratic solution we obtain:

$$n_I^2 = \frac{1 \pm \sqrt{1 + 4 \left( \frac{\sigma}{2 \varepsilon_0 \omega} \right)^2}}{2}$$

$$n_I^2 = \frac{1 + \sqrt{1 + 4 \left( \frac{\sigma}{2 \varepsilon_0 \omega} \right)^2}}{2}$$

We need to take the positive root because $n_I$ is a real number.
For most metals, with light in the microwave region, \( \frac{\sigma}{\omega} \gg \varepsilon_0 \)

With this approximation, the complex refractive index for metals becomes

\[
\begin{align*}
n_i^2 &\approx \frac{\sigma}{2 \varepsilon_0 \omega} \quad \Rightarrow \quad n_i = \sqrt{\frac{\sigma}{2 \varepsilon_0 \omega}} \quad n_R = \frac{\sigma}{2 n_i \varepsilon_0 \omega} = \sqrt{\frac{\sigma}{2 \varepsilon_0 \omega}}
\end{align*}
\]

Substituting our expression for the complex refractive index back into our expression for the electric field we obtain

\[
\begin{align*}
\tilde{E} &= \tilde{E}_0 \exp \left[ i \left( \tilde{k} \cdot \tilde{r} - \omega t \right) \right] = \tilde{E}_0 \exp \left\{ i \left[ (n_R + i n_i) \frac{\omega}{c} (\hat{u}_k \cdot \tilde{r}) - \omega t \right] \right\} \\
&= \tilde{E}_0 \exp \left\{ i \omega \left[ \frac{n_R}{c} (\hat{u}_k \cdot \tilde{r}) - t \right] \right\} \exp \left[ - \frac{n_i \omega}{c} (\hat{u}_k \cdot \tilde{r}) \right]
\end{align*}
\]

The first exponential term is oscillatory. The EM wave propagates with a velocity of \( n_R / c \).
The second exponential has a real argument (absorbed).
The second term leads to absorption of the beam in metals due to inducing a current in the medium. This causes the irradiance to decrease as the wave propagates through the medium.

\[ I \equiv \tilde{E}\tilde{E}^* = \tilde{E}_0\tilde{E}_0^* \exp \left[ - \frac{2n_I \omega (\hat{u}_k \cdot \vec{r})}{c} \right] \]

\[ I = I_0 \exp \left[ - \frac{2n_I \omega (\hat{u}_k \cdot \vec{r})}{c} \right] = I_0 \exp \left[ - \alpha (\hat{u}_k \cdot \vec{r}) \right] \]

The absorption coefficient is defined: \[ \alpha = \frac{2n_I \omega}{c} = \frac{4\pi n_I}{\lambda} \]
20-7. Reflection from Metals

Reflection from metals is analyzed by substituting the complex refractive index $\tilde{n}$ in the Fresnel equations:

**TE case:**
$$ r = \frac{E_r}{E_i} = \frac{\cos \theta_i - \sqrt{n^2 - \sin^2 \theta_i}}{\cos \theta_i + \sqrt{n^2 - \sin^2 \theta_i}} $$

**TM case:**
$$ r = \frac{E_r}{E_i} = \frac{\tilde{n}^2 \cos \theta_i - \sqrt{\tilde{n}^2 - \sin^2 \theta_i}}{\tilde{n}^2 \cos \theta_i + \sqrt{\tilde{n}^2 - \sin^2 \theta_i}} $$

Substituting $\tilde{n} = n_R + i n_I$ we obtain:

**TE case:**
$$ r = \frac{E_r}{E_i} = \frac{\cos \theta_i - \sqrt{(n_R^2 - n_I^2 - \sin^2 \theta_i) + i(2n_R n_I)}}{\cos \theta_i + \sqrt{(n_R^2 - n_I^2 - \sin^2 \theta_i) + i(2n_R n_I)}} $$

**TM case:**
$$ r = \frac{E_r}{E_i} = \frac{\left[(n_R^2 - n_I^2) + i(2n_R n_I)\right] \cos \theta_i - \sqrt{(n_R^2 - n_I^2 - \sin^2 \theta_i) + i(2n_R n_I)}}{\left[(n_R^2 - n_I^2) + i(2n_R n_I)\right] \cos \theta_i + \sqrt{(n_R^2 - n_I^2 - \sin^2 \theta_i) + i(2n_R n_I)}} $$
Reflection from Metals at normal incidence ($\theta_i=0^\circ$)

At normal incidence, $\theta_i = 0^\circ$:

**TE & TM cases**:

\[
r = \frac{E_r}{E_i} = \frac{\cos \theta_i - \sqrt{n_R^2 - n_i^2 - \sin^2 \theta_i} + i(2n_R n_i)}{\cos \theta_i + \sqrt{n_R^2 - n_i^2 - \sin^2 \theta_i} + i(2n_R n_i)}
\]

\[
= \frac{1 - \sqrt{(n_R^2 - n_i^2)} + i(2n_R n_i)}{1 + \sqrt{(n_R^2 - n_i^2)} + i(2n_R n_i)} = \frac{1 - \sqrt{(n_R - i n_i)^2}}{1 + \sqrt{(n_R - i n_i)^2}}
\]

\[
\therefore \quad r = \frac{1 - (n_R - i n_i)}{1 + (n_R - i n_i)}
\]

**The power reflectance $R$ is given by**

\[
R = r r^* = \left[ \frac{1 - (n_R - i n_i)}{1 + (n_R - i n_i)} \right] \left[ \frac{1 - (n_R + i n_i)}{1 + (n_R + i n_i)} \right] = \left( \frac{1 - 2n_R + n_R^2 + n_i^2}{1 + 2n_R + n_R^2 + n_i^2} \right)
\]

\[
R = \frac{(n_R - 1)^2 + n_i^2}{(n_R + 1)^2 + n_i^2}
\]
Reflection from Metals

Reflectance

At normal incidence
(from Hecht, page 113)

Figure 20-10  Reflectance from metal surfaces by using Fresnel’s equations. The values of $n_0$ and $n_i$ are given for sodium light of $\lambda = 589.3$ nm.

Figure 4.42  Reflectance versus wavelength for silver, gold, copper, and aluminum.