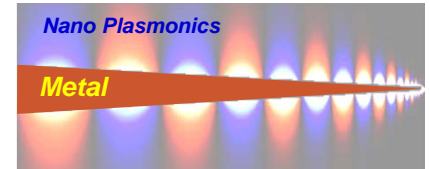


# Nanophotonics



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Professor : Seok Ho Song, [shsong@hanyang.ac.kr](mailto:shsong@hanyang.ac.kr), 02-2220-0923 (Room# 36-401)

Textbook : 1. Stefan Maier, "Plasmonics: Fundamentals and Applications", Springer  
2. Keiko Iizuka, "Elements of photonics", John Wiley & Sons, Inc.  
3. Joseph W. Goodman, "Introduction to Fourier optics", McGraw-Hill Co.

Evaluation : Att 20%, Homework 20%, Mid-term 30%, Final 30%

## Introduction and basics

Introduction to Nanophotonics ([pdf](#))

Nature of diffraction ([pdf](#))

Fourier analysis in linear systems (Goodman 2.1) ([pdf](#))

Plane waves and spatial frequency (Iizuka 1.1) ([pdf](#))

Angular spectra in beam propagation ([pdf](#))

Optical properties of materials: dielectrics and metals ([pdf](#))

## Surface plasmon-polaritons

Dispersion relation of single metal-dielectric interfaces ([pdf](#))

Surface plasmon excitation ([pdf](#))

EM energy density in metals ([pdf](#))

Localized particle plasmons – Rayleigh & Mie scattering ([pdf](#))

Dispersion relation of metal nanorods and nanotips ([pdf](#))

Dispersion relation of SPPs on thin metal films ([pdf](#))

Dielectric loaded SPP waveguides ([pdf](#))

SPP waveguides ([pdf](#))

Metal-Insulator-Metal plasmonic slot-waveguides ([pdf](#))

SPP waveguide sensors ([pdf](#))

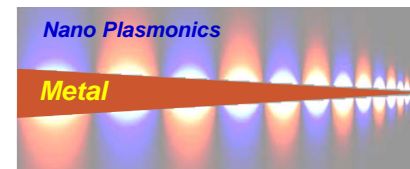
Surface-plasmon mediated light sources ([pdf](#))

SPP Photovoltaic ([pdf](#))

Summary-Nanoplasmonics-Enhancement of optical processes is severely limited by the metal loss ([pdf](#))

# Nanophotonics

Nanophotonics, Paras N. Prasad, 2004, John Wiley & Sons, Inc., Hoboken, New Jersey., ISBN 0-471-64988-0



## Photons and Electrons

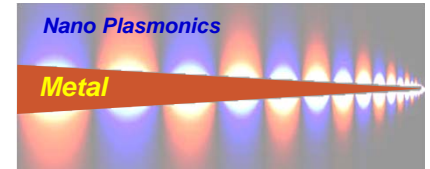
Both photons and electrons are elementary particles that simultaneously exhibit particle and wave-type behavior.

Photons and electrons may appear to be quite different as described by classical physics, which defines photons as electromagnetic waves transporting energy and electrons as the fundamental charged particle (lowest mass) of matter.

A quantum description, on the other hand, reveals that photons and electrons can be treated analogously and exhibit many similar characteristics.

**Table 2.1.** Similarities in Characteristics of Photons and Electrons

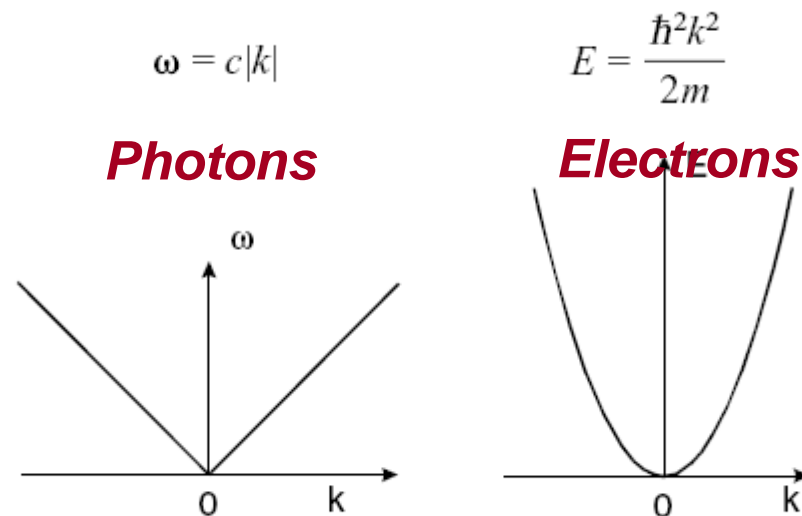
| Photons   | Electrons   |
|---|---|
| <b>Wavelength</b>   |   |
| $\lambda = \frac{h}{p} = \frac{c}{\nu}$   | $\lambda = \frac{h}{p} = \frac{h}{mv}$  |
| <b>Eigenvalue (Wave) Equation</b>   |   |
| $\left\{ \nabla \times \frac{1}{\epsilon(r)} \nabla \times \right\} \mathbf{B}(r) = \left( \frac{\omega}{c} \right)^2 \mathbf{B}(r)$  | $\hat{H}\psi(r) = -\frac{\hbar^2}{2m}(\nabla \cdot \nabla + V(r))\psi(r) = E\psi$   |
| <b>Free-Space Propagation</b>   |   |
| Plane wave<br>$\mathbf{E} = \left( \frac{1}{2} \right) \mathbf{E}^0 (e^{i\mathbf{k} \cdot \mathbf{r} - \omega t} + e^{-i\mathbf{k} \cdot \mathbf{r} + \omega t})$<br>$\mathbf{k}$ = wavevector, a real quantity | Plane wave:<br>$\Psi = c(e^{i\mathbf{k} \cdot \mathbf{r} - \omega t} + e^{-i\mathbf{k} \cdot \mathbf{r} + \omega t})$<br>$\mathbf{k}$ = wavevector, a real quantity |
| <b>Interaction Potential in a Medium</b>  |   |
| Dielectric constant (refractive index)  | Coulomb interactions  |
| <b>Propagation Through a Classically Forbidden Zone</b>   |   |
| Photon tunneling (evanescent wave) with wavevector, $\mathbf{k}$ , imaginary and hence amplitude decaying exponentially in the forbidden zone   | Electron-tunneling with the amplitude (probability) decaying exponentially in the forbidden zone  |
| <b>Localization</b>   |   |
| Strong scattering derived from large variations in dielectric constant (e.g., in photonic crystals)   | Strong scattering derived from a large variation in Coulomb interactions (e.g., in electronic semiconductor crystals)   |
| <b>Cooperative Effects</b>  |   |
| Nonlinear optical interactions  | Many-body correlation<br>Superconducting Cooper pairs<br>Biexciton formation  |



# Free-Space Propagation of photons and electrons

In a “free-space” propagation, there is no interaction potential or it is constant in space. For photons, it simply implies that no spatial variation of refractive index  $n$  occurs.

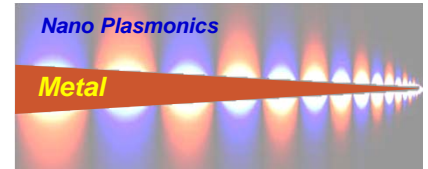
The wavevector dependence of energy is different for photons (linear dependence) and electrons (quadratic dependence).



**Figure 2.1.** Dispersion relation showing the dependence of energy on the wavevector for a free-space propagation. (a) Dispersion for photons. (b) Dispersion for electrons.

## Band structure (Dispersion relation)

For free-space propagation, all values of frequency for photons and energy  $E$  for electrons are permitted. This set of allowed continuous values of frequency (or energy) form together a band, and the **band structure** refers to the characteristics of the dependence of the frequency (or energy) on the wavevector  $k$ .



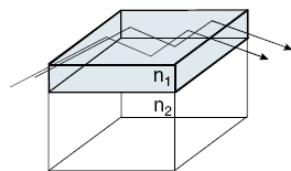
# Confinement of Photons and Electrons

In the case of photons, the confinement can be introduced by trapping light in a region of high refractive index or with high surface reflectivity.

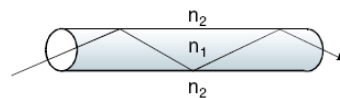
The confinement of electrons also leads to modification of their wave properties and produces quantization—that is, discrete values for the possible eigenmodes.

## Photons

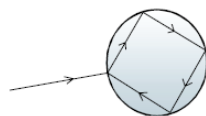
Confinement of Photon



Optical planar waveguide



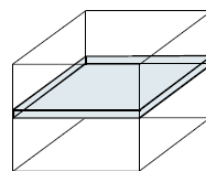
Optical fiber



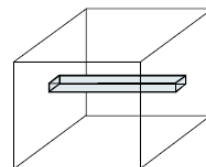
Microsphere optical cavity

## Electrons

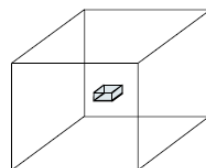
Confinement of Electron



Quantum well



Quantum wire



Quantum dot

The field distribution and the corresponding propagation constant are obtained by the solution of the **Maxwell's equation** and imposing the boundary conditions (defining the boundaries of the waveguide and the refractive index contrast). The solution of the wave equation shows that the confinement produces certain discrete sets of field distributions called *eigenmodes*, which are labeled by *quantum numbers* (integer).

The corresponding wave equation for electrons is the **Schrödinger equation**. The potential confining the electron is the energy barrier—that is, regions where the potential energy  $V$  is much higher than the energy  $E$  of the electron.

$$\mathbf{E} = \frac{1}{2} f(x, y) a(z) (e^{i\beta z} + e^{-i\beta z}) \quad \Psi_n(x) = \frac{1}{2i} \left( \frac{2}{l} \right)^{1/2} (e^{ikx} - e^{-ikx})$$

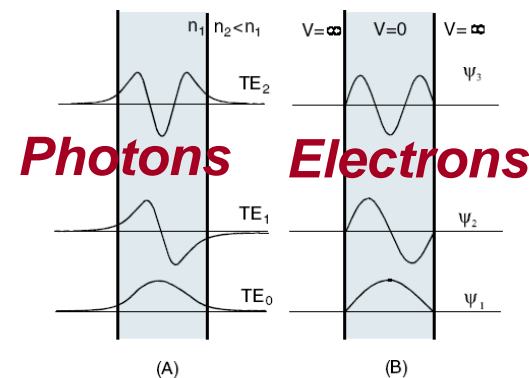
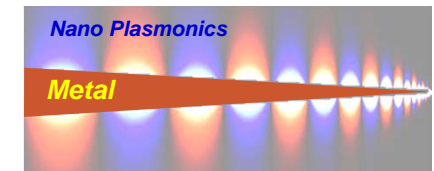


Figure 2.3. (A) Electric field distribution for TE modes  $n = 0, 1, 2$  in a planar waveguide with one-dimensional confinement of photons. (B) Wavefunction  $\psi$  for quantum levels  $n = 1, 2, 3$  for an electron in a one-dimensional box.

# MANIFESTATIONS OF QUANTUM CONFINEMENT



**Size Dependence of Optical Properties.** Quantum confinement produces a blue shift in the bandgap as well as appearance of discrete subbands corresponding to quantization along the direction of confinement.

As the dimensions of confinement increase, the bandgap decreases; hence the interband transitions shift to longer wavelengths, finally approaching the bulk value for a large width.

**Increase of Oscillator Strength.** Quantum confinement produces a major modification in the density of states both for valence and conduction bands. The oscillator strength of an optical transition for an interband transition depends on the joint density of states of the levels in the valence band and the levels in the conduction bands, between which the optical transition occurs.

**New Intraband Transitions.** In quantum-confined structures, there are sub-bands characterized by the different quantum numbers ( $n = 1, 2, \dots$ ). These new transitions are in IR and have been utilized to produce inter sub-band detectors and lasers, the most interesting of which are quantum cascade lasers

**Increased Exciton Binding.** Quantum confinement of electrons and holes also leads to enhanced binding between them and thereby produces increased exciton binding energy. Thus, excitonic resonances are very pronounced in quantum-confined structures and, in the strong confinement conditions, can be seen even at room temperature.

**Increase of Transition Probability in Indirect Gap Semiconductor.** In the quantum-confined structures, confinement of electrons produces a reduced uncertainty  $\Delta x$  in its position and, consequently, produces a larger uncertainty  $\Delta k$  in its quasi-momentum. Confinement, therefore, relaxes the quasi-momentum  $\Delta k$  selection rule, thus allowing enhanced emission to be observed in porous silicon and silicon nanoparticles.

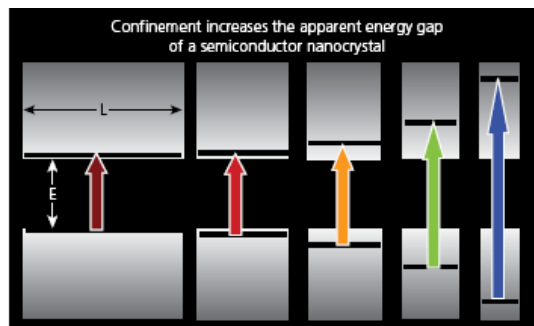
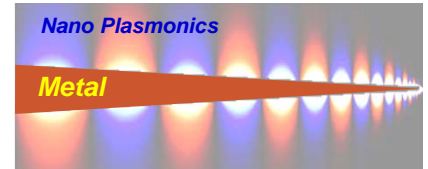


Figure 2. Schematic of the effect of the decreased size of the box on the increased energy gap of a semiconductor quantum dot, and the resultant luminescent color change from bulk materials (left) to small nanocrystals (right).

Table 4.2. Optics of Quantum Confined Semiconductors

| Optical Transitions                                      |  |                            |  |
|--|--|----------------------------|--|
| Absorption   |  | Luminescence               |  |
| Interband:   | Intraband (Inter-sub-band):  | Photoluminescence:         | Electroluminescence:   |
| Transition between modified valence and conduction bands | Transition between quantized sub-bands of a band (e.g., conduction band) | Optically excited emission | Emission generated by recombination of electrically injected electrons and holes |



# Propagation Through a Classically Forbidden Zone: Tunneling

In a classical picture, the photons and electrons are completely confined in the regions of confinement.

For photons, it is seen by the ray optics for the propagating wave.

Similarly, classical physics predicts that, once trapped within the potential energy barriers where the energy  $E$  of an electron is less than the potential energy  $V$  due to the barrier, the electron will remain completely confined within the walls.

However, the wave picture does not predict so.

The field distribution of light confined in a waveguide extends beyond the boundaries of the waveguide.

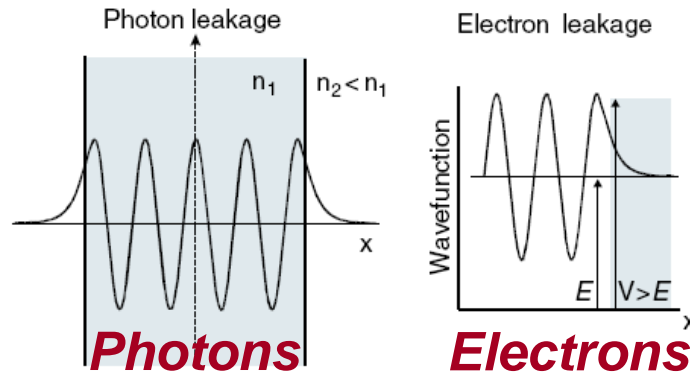


Figure 2.5. Schematic representation of leakage of photons and electrons into classically energetically forbidden regions.

This light leakage generates an electromagnetic field called evanescent wave.

$$E_x = E_0 \exp(-x/d_p)$$

In an analogous fashion, an electron shows a leakage through regions where  $E < V$ . The wavefunction extending beyond the box into the region of  $V > E$  decays exponentially, just like the evanescent wave for confined light. The transmission probability is

$$T = ae^{-2kl} \quad k \text{ is equal to } (2mE)^{1/2}/\hbar.$$

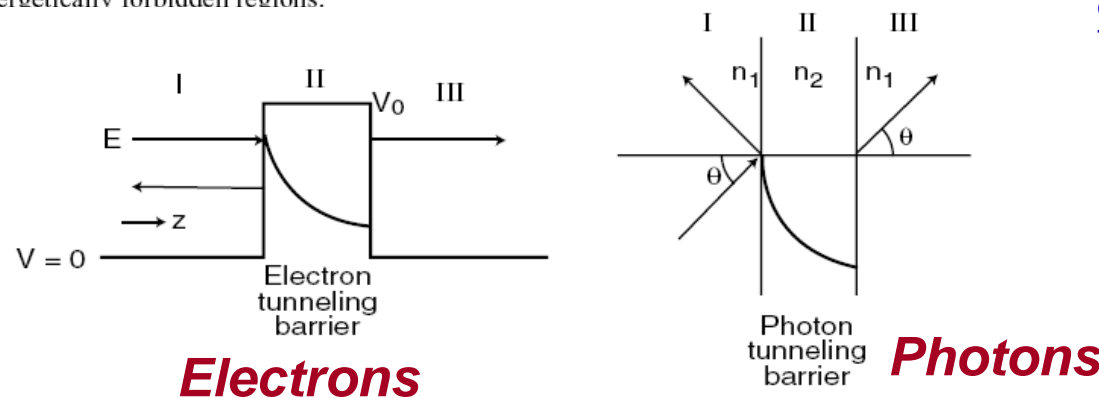
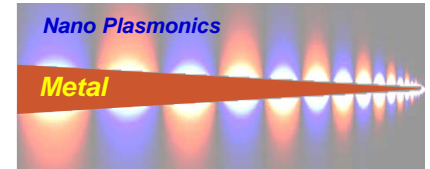
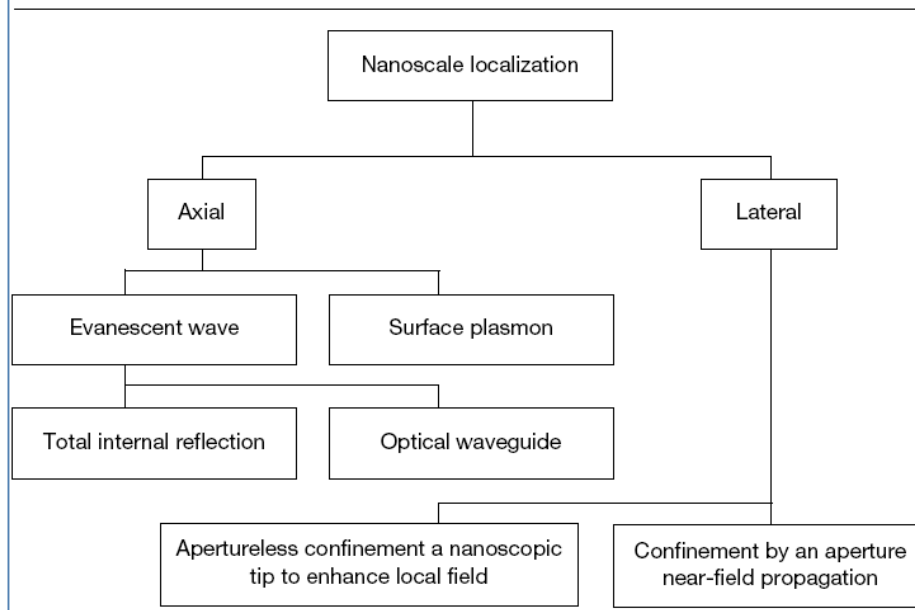


Figure 2.6. Schematics of electron and photon tunneling through a barrier.

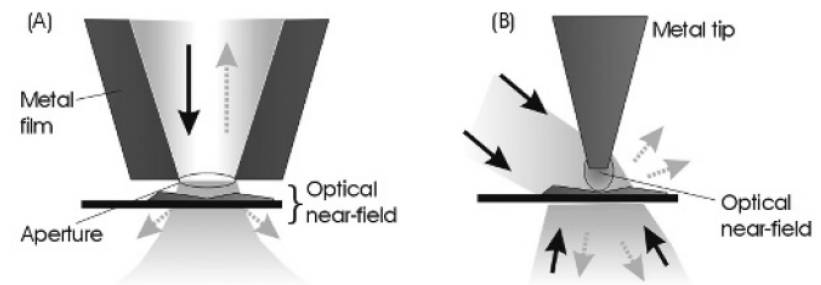
# NANOSCALE OPTICAL INTERACTIONS



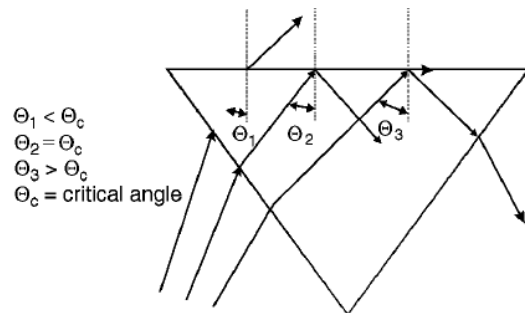
**Table 2.2.** Methods for Nanoscale Localization of Electromagnetic Field



## Lateral Nanoscopic Localization – NSOM, SNOM



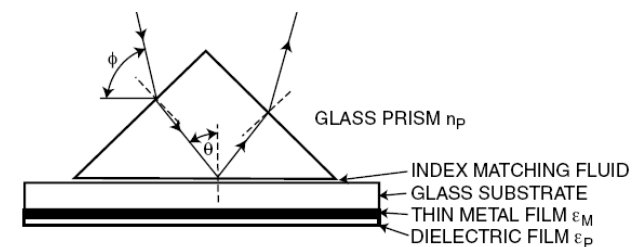
## Axial Nanoscopic Localization - Evanescent Wave



The penetration depths  $d_p$  for the visible light are 50–100 nm.

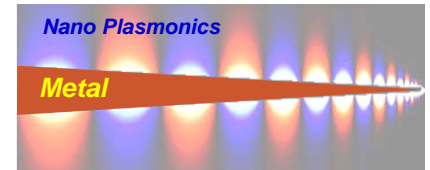
$$d_p = \lambda / [4\pi n_1 \{ \sin^2 \theta - (n_2/n_1)^2 \}^{1/2}]$$

## Axial Nanoscopic Localization - Surface Plasmon Resonance



$$k_{sp} = (\omega/c) [(\epsilon_m \epsilon_d) / (\epsilon_m + \epsilon_d)]^{1/2}$$

# Localization Under a Periodic Potential: Bandgap



Both photons and electrons show an analogous behavior when subjected to a periodic potential.

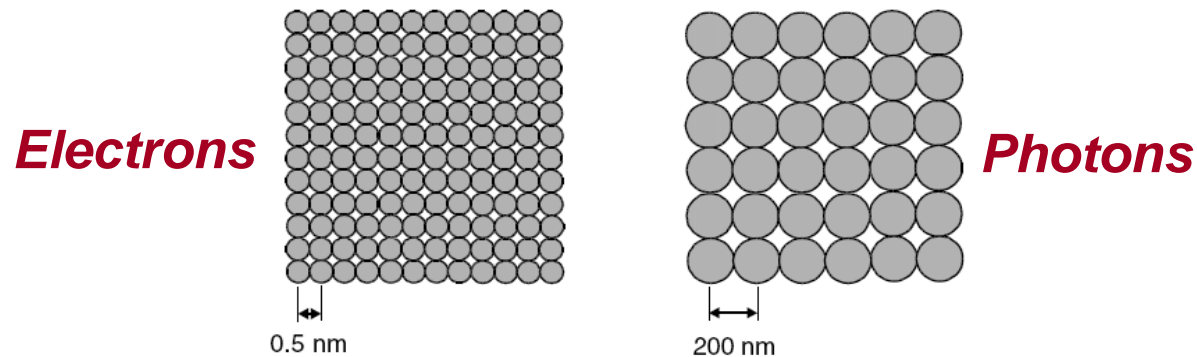


Figure 2.7. Schematic representation of an electronic crystal (left) and a photonic crystal (right).

The solution of the Schrödinger equation for the energy of electrons, now subjected to the periodic potential  $V$ , produces a splitting of the electronic band: the lower energy band is called the **valence band**, the higher energy band is called the **conduction band**. These two bands are separated by a “forbidden” energy gap, the width of which is called the **bandgap**.

In the case of a photonic crystal, the eigenvalue equation for photons can be used to calculate the dispersion relation  $\omega$  versus  $k$ . A similar type of band splitting is observed for a photonic crystal, and a forbidden frequency region exists between the two bands, similar to that between the valence and the conduction band of an electronic crystal, which is often called the **photonic bandgap**.

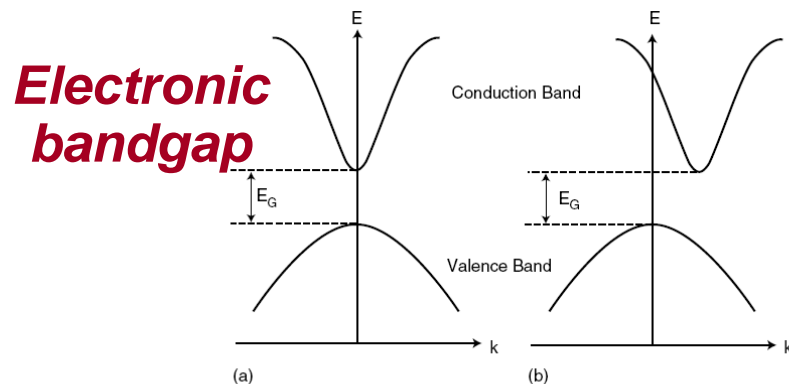


Figure 2.8. Schematics of electron energy in (a) direct bandgap (e.g., GaAs, InP, CdS) and (b) indirect bandgap (e.g., Si, Ge, GaP) semiconductors.

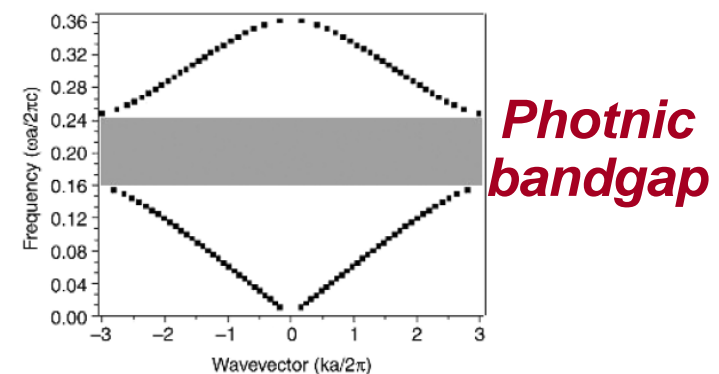


Figure 2.9. Dispersion curve for a one-dimensional photonic crystal showing the lowest energy bandgap.

# Some basics

Maxwell's Equations

Matrix Theory of Multilayer Optics: Transfer Matrix and Scattering Matrix

# Maxwell's Equations

(In Free Space) Maxwell's Equations

$$\begin{array}{l}
 \nabla \times \mathcal{H} = \epsilon_o \frac{\partial \mathcal{E}}{\partial t} \\
 \nabla \times \mathcal{E} = -\mu_o \frac{\partial \mathcal{H}}{\partial t} \\
 \nabla \cdot \mathcal{E} = 0 \\
 \nabla \cdot \mathcal{H} = 0,
 \end{array}
 \xrightarrow{\nabla \times (\nabla \times \mathcal{E}) = \nabla(\nabla \cdot \mathcal{E}) - \nabla^2 \mathcal{E}}
 \begin{array}{l}
 \nabla^2 u - \frac{1}{c_o^2} \frac{\partial^2 u}{\partial t^2} = 0 \\
 c_o = \frac{1}{\sqrt{\epsilon_o \mu_o}}
 \end{array}$$

(In a medium) Maxwell's Equations

$$\begin{array}{l}
 \nabla \times \mathcal{H} = \frac{\partial \mathcal{D}}{\partial t} \\
 \nabla \times \mathcal{E} = -\frac{\partial \mathcal{B}}{\partial t} \\
 \nabla \cdot \mathcal{D} = 0 \\
 \nabla \cdot \mathcal{B} = 0.
 \end{array}$$

→ (In a Linear, Nondispersive, Homogeneous, and Isotropic Medium)

$$\begin{array}{l}
 \mathcal{D} = \epsilon_o \mathcal{E} + \mathcal{P} \\
 \mathcal{B} = \mu_o \mathcal{H} + \mu_o \mathcal{M} \\
 \mathcal{P} = \epsilon_o \chi \mathcal{E} \\
 \mathcal{D} = \epsilon \mathcal{E} \quad \epsilon = \epsilon_o (1 + \chi) \\
 \mathcal{B} = \mu \mathcal{H}
 \end{array}$$

$$\begin{array}{l}
 \nabla \times \mathcal{H} = \epsilon \frac{\partial \mathcal{E}}{\partial t} \\
 \nabla \times \mathcal{E} = -\mu \frac{\partial \mathcal{H}}{\partial t} \\
 \nabla \cdot \mathcal{E} = 0 \\
 \nabla \cdot \mathcal{H} = 0.
 \end{array}$$

$$\begin{array}{l}
 \nabla^2 u - \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} = 0 \\
 c = \frac{1}{\sqrt{\epsilon \mu}}
 \end{array}$$

$$n = \frac{c_o}{c} = \sqrt{\frac{\epsilon}{\epsilon_o} \frac{\mu}{\mu_o}} \quad \mu = \mu_o \quad n = \sqrt{\frac{\epsilon}{\epsilon_o}} = \sqrt{1 + \chi}$$

# Maxwell's Equations

(In a inhomogeneous Medium)  $\rightarrow \epsilon = \epsilon(\mathbf{r}) \rightarrow$  position dependent

$$\begin{aligned}
 \nabla \times \mathcal{H} &= \frac{\partial \mathcal{D}}{\partial t} \longrightarrow \nabla \times \left( \frac{\epsilon_o}{\epsilon} \nabla \times \mathcal{H} \right) = -\frac{1}{c_o^2} \frac{\partial^2 \mathcal{H}}{\partial t^2} \\
 \nabla \times \mathcal{E} &= -\frac{\partial \mathcal{B}}{\partial t} \longrightarrow \frac{\epsilon_o}{\epsilon} \nabla \times (\nabla \times \mathcal{E}) = -\frac{1}{c_o^2} \frac{\partial^2 \mathcal{E}}{\partial t^2} \\
 \nabla \cdot \mathcal{D} &= 0 \\
 \nabla \cdot \mathcal{B} &= 0.
 \end{aligned}$$

$\nabla \times (\nabla \times \mathcal{E}) = \nabla(\nabla \cdot \mathcal{E}) - \nabla^2 \mathcal{E} \longrightarrow \nabla^2 \mathcal{E} + \nabla \left( \frac{1}{\epsilon} \nabla \epsilon \cdot \mathcal{E} \right) - \mu_o \epsilon \frac{\partial^2 \mathcal{E}}{\partial t^2} = 0$

$\nabla \cdot \epsilon \mathcal{E} = 0$   
 $\nabla \cdot \epsilon \mathcal{E} = \epsilon \nabla \cdot \mathcal{E} + \nabla \epsilon \cdot \mathcal{E},$   
 $\nabla \cdot \mathcal{E} = -(1/\epsilon) \nabla \epsilon \cdot \mathcal{E}$

(In an Anisotropic Medium)  $\rightarrow \mathcal{P}_i = \sum_j \epsilon_o \chi_{ij} \mathcal{E}_j$

(In a Dispersive Medium)  $\rightarrow \mathcal{P}(t) = \epsilon_o \int_{-\infty}^{\infty} \mathbf{x}(t-t') \mathcal{E}(t') dt'$

$\epsilon_o \mathbf{x}(t) \rightarrow$  impulse response function

$\epsilon_o \chi(\nu) \rightarrow$  transfer function = FT of  $\chi(t)$

$$\chi(\nu) = \int_{-\infty}^{\infty} \mathbf{x}(t) \exp(-j2\pi\nu t) dt$$

# Maxwell's Equations

## For monochromatic electromagnetic waves

(In a source-free Medium)  $\rightarrow$

$$\begin{array}{lcl}
 \nabla \times \mathcal{H} = \frac{\partial \mathcal{D}}{\partial t} & & \\
 \nabla \times \mathcal{E} = -\frac{\partial \mathcal{B}}{\partial t} & \xrightarrow{\begin{array}{l} \mathcal{E}(\mathbf{r}, t) = \text{Re}\{\mathbf{E}(\mathbf{r}) \exp(j\omega t)\} \\ \mathcal{H}(\mathbf{r}, t) = \text{Re}\{\mathbf{H}(\mathbf{r}) \exp(j\omega t)\} \end{array}} & \begin{array}{l} \nabla \times \mathbf{H} = j\omega \mathbf{D} \\ \nabla \times \mathbf{E} = -j\omega \mathbf{B} \\ \nabla \cdot \mathbf{D} = 0 \\ \nabla \cdot \mathbf{B} = 0. \end{array} \\
 \nabla \cdot \mathcal{D} = 0 & & \\
 \nabla \cdot \mathcal{B} = 0. & & 
 \end{array}$$

(In a Linear, Nondispersive, Homogeneous, source-free, and Isotropic Medium)

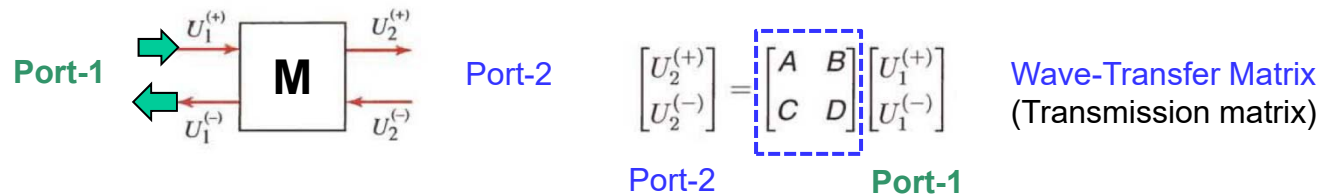
$$\begin{array}{lcl}
 \nabla \times \mathbf{H} = j\omega \epsilon \mathbf{E} & & \\
 \nabla \times \mathbf{E} = -j\omega \mu \mathbf{H} & \xrightarrow{\begin{array}{l} \mathbf{D} = \epsilon \mathbf{E} \\ \mathbf{B} = \mu \mathbf{H} \end{array}} & \nabla^2 U + k^2 U = 0 \rightarrow \text{Helmholtz Equation} \\
 \nabla \cdot \mathbf{E} = 0 & & k = nk_o = \omega \sqrt{\epsilon \mu} \\
 \nabla \cdot \mathbf{H} = 0. & & k_o = \omega / c_o
 \end{array}$$

(In a Inhomogeneous medium)  $\rightarrow$  Generalized Helmholtz Equations

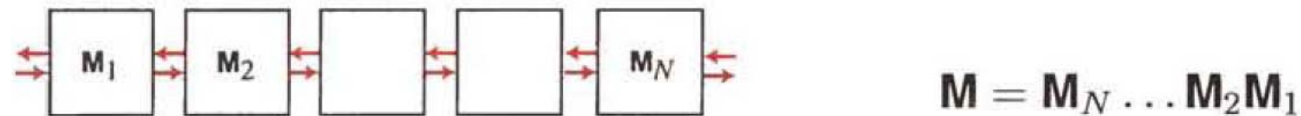
$$\begin{array}{l}
 \eta(\mathbf{r}) \nabla \times (\nabla \times \mathbf{E}) = \frac{\omega^2}{c_o^2} \mathbf{E}, \\
 \nabla \times [\eta(\mathbf{r}) \nabla \times \mathbf{H}] = \frac{\omega^2}{c_o^2} \mathbf{H},
 \end{array}
 \quad \eta(\mathbf{r}) = \epsilon_o / \epsilon(\mathbf{r})$$

# Matrix Theory of Multilayer Optics: Transfer Matrix and Scattering Matrix

**Wave-Transfer Matrix:** the complex amplitudes of the forward and backward waves through the boundaries (PORTs) of a multilayered medium

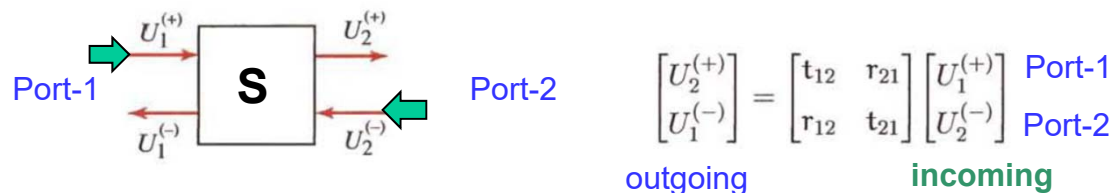


For a multilayered medium,



→ Identical to the ray transfer (ABCD) matrix

**Scattering Matrix:** the outgoing waves are expressed in terms of the incoming waves



$t_{ij}, r_{ij}$ : transmission and reflection coefficients from port- $i$  to port- $j$

→ The S matrix of a cascade of elements is **not the product of the S matrices** of the constituent elements.

## Relation between Scattering Matrix and Wave-Transfer Matrix

$$\mathbf{M} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \frac{1}{t_{21}} \begin{bmatrix} t_{12}t_{21} - r_{12}r_{21} & r_{21} \\ -r_{12} & 1 \end{bmatrix}, \quad (7.1-5)$$


$$\mathbf{S} = \begin{bmatrix} t_{12} & r_{21} \\ r_{12} & t_{21} \end{bmatrix} = \frac{1}{D} \begin{bmatrix} AD - BC & B \\ -C & 1 \end{bmatrix}. \quad (7.1-6)$$

Conversion Relations

※ These equations are **not valid** in the limiting cases **when**  $t_{21} = 0$  or  $D = 0$ .

## Two Cascaded Systems

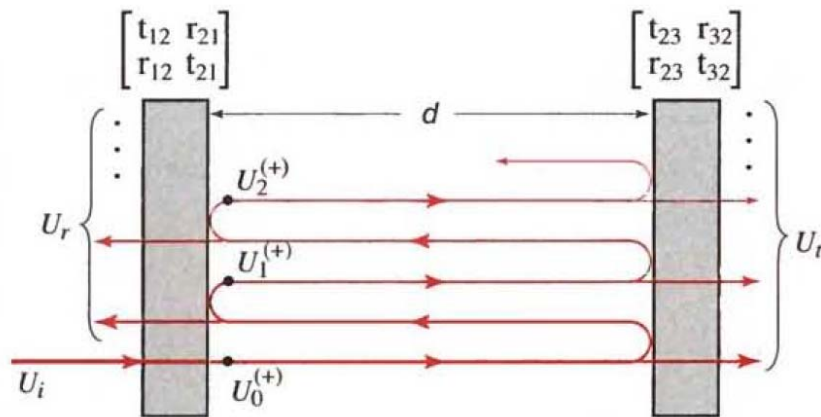
|  |  |
|--|--|
| $\begin{bmatrix} t_{12} & r_{21} \\ r_{12} & t_{21} \end{bmatrix}$ | $\begin{bmatrix} t_{23} & r_{32} \\ r_{23} & t_{32} \end{bmatrix}$ |
|--|--|



|   |
|---|
| $t_{13} = \frac{t_{12}t_{23}}{1 - r_{21}r_{23}}, \quad r_{13} = r_{12} + \frac{t_{12}t_{21}r_{23}}{1 - r_{21}r_{23}}$ |
|---|

## Two Cascaded Systems: *Airy's Formulas*

If the two cascaded systems are mediated by propagation through a homogeneous medium,

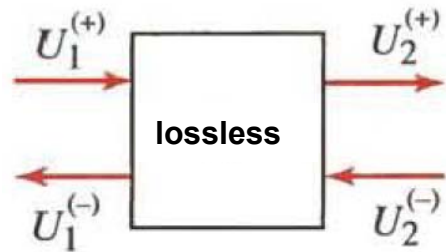


$$t_{13} = \frac{t_{12}t_{23} \exp(-j\varphi)}{1 - r_{21}r_{23} \exp(-j2\varphi)}, \quad r_{13} = r_{12} + \frac{t_{12}t_{21}r_{23} \exp(-j2\varphi)}{1 - r_{21}r_{23} \exp(-j2\varphi)}.$$

(7.1-8)  
Airy's  
Formulas

where,  $\varphi = nk_o d$ .

## Conservation Relations for Lossless Media



The incoming and outgoing optical powers must be equal.

$$\Rightarrow |U_1^{(+)}|^2 + |U_2^{(-)}|^2 = |U_2^{(+)}|^2 + |U_1^{(-)}|^2$$

If the media at the input and output planes have the same impedance and refractive index,

By choosing the incoming amplitudes  $U_1^{(+)}$  and  $U_2^{(-)}$  to be (1,0), (0,1), and (1,1),

$$\begin{bmatrix} U_2^{(+)} \\ U_1^{(-)} \end{bmatrix} = \begin{bmatrix} t_{12} & r_{21} \\ r_{12} & t_{21} \end{bmatrix} \begin{bmatrix} U_1^{(+)} \\ U_2^{(-)} \end{bmatrix}$$



$$|t_{12}| = |t_{21}| \equiv |t|, \quad |r_{12}| = |r_{21}| \equiv |r|, \quad |t|^2 + |r|^2 = 1, \\ t_{12}/t_{21}^* = -r_{12}/r_{21}^*.$$

From the conversion relation  
Between  $\mathbf{M}$  and  $\mathbf{S}$



$$|D| = |A|, \quad |C| = |B|, \quad |A|^2 - |B|^2 = 1, \\ \det \mathbf{M} = C/B^* = A/D^* = t_{12}/t_{21}, \quad |\det \mathbf{M}| = 1.$$

## Lossless Reciprocal Systems

For lossless systems with reciprocal symmetry, namely systems whose transmission/reflection in the forward and backward directions are identical, we have  $t_{21} = t_{12} \equiv t$  and  $r_{21} = r_{12} \equiv r$ .

→  $|t|^2 + |r|^2 = 1, \quad t/r = -(t/r)^*, \quad \arg\{t\} - \arg\{r\} = \pm\pi/2$   
 $A = D^*, \quad B = C^*, \quad |A|^2 - |B|^2 = 1, \quad \det \mathbf{M} = 1$



$$\mathbf{S} = \begin{bmatrix} t & r \\ r & t \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} 1/t^* & r/t \\ r^*/t^* & 1/t \end{bmatrix},$$

(7.1-15)  
Lossless Reciprocal System

# ONE-DIMENSIONAL PHOTONIC CRYSTALS

From the Generalized Helmholtz equation,

$$\eta(\mathbf{r}) \nabla \times (\nabla \times \mathbf{E}) = \frac{\omega^2}{c_o^2} \mathbf{E},$$

$$\nabla \times [\eta(\mathbf{r}) \nabla \times \mathbf{H}] = \frac{\omega^2}{c_o^2} \mathbf{H},$$

One-dimensional (1D) photonic crystals are dielectric structures whose optical properties vary periodically in one direction,

$$\eta(z + \Lambda) = \eta(z) = \epsilon_o / \epsilon(z)$$

For an ON-AXIS wave traveling along the z axis and polarized in the X direction,

$$\nabla \times [\eta(\mathbf{r}) \nabla \times \mathbf{H}] = \frac{\omega^2}{c_o^2} \mathbf{H} \longrightarrow -\frac{d}{dz} \left[ \eta(z) \frac{d}{dz} \right] H_y = \frac{\omega^2}{c_o^2} H_y$$

For an OFF-AXIS wave, i.e., a wave traveling in an arbitrary direction in the x-z plane,

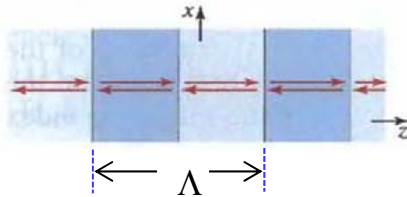
$$\nabla \times [\eta(\mathbf{r}) \nabla \times \mathbf{H}] = \frac{\omega^2}{c_o^2} \mathbf{H} \longrightarrow \left\{ -\frac{\partial}{\partial z} \left[ \eta(z) \frac{\partial}{\partial z} \right] + \eta(z) \frac{\partial^2}{\partial x^2} \right\} H_y = \frac{\omega^2}{c_o^2} H_y$$

for TM off-axis wave

**➔ Before embarking on finding solutions to these eigenvalue problems, we first examine the conditions imposed on the propagating modes by the translational symmetry associated with the periodicity.**

# Bloch Mode (Block Wave)

## On-Axis Bloch Modes



1D periodic medium is **invariant to translation** by the distance  $\Lambda$  along the axis of periodicity.

→ Optical modes must have the form

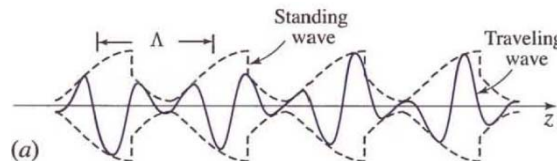
$$U(z) = p_K(z) \exp(-jKz) \rightarrow \text{Bloch mode (wave)}$$

→  $p_K(z)$  is a periodic function of period  $\Lambda$

→  $K$ : **Bloch wavenumber**

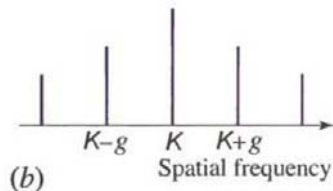
→ The **wave** is **unaltered by a translation  $\Lambda$**  and altered only a phase factor  $\exp(-jK\Lambda)$

The Bloch mode is thus a **plane wave  $\exp(-jKz)$**  with propagation constant  $K$ , **modulated by a periodic function  $p_K(z)$** , which has the character of a standing wave.



Since a periodic function of period  $\Lambda$  can be expanded in a Fourier series as a superposition of harmonic functions of the form  $\exp(-jmgz)$ ,  $m = 0, \pm 1, \pm 2, \dots$ , with  $g = 2\pi/\Lambda$

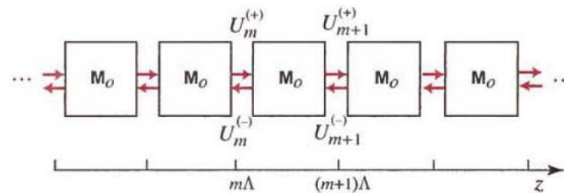
Bloch wave is a **superposition of plane waves of multiple spatial frequencies  $K+mg$** .



For a **complete specification of all modes**, we need only consider values of  $K$  in a spatial-frequency interval of width  $g=2\pi/\Lambda$ . Interval  $[-g/2, g/2] = [-\pi/\Lambda, \pi/\Lambda]$  : → **First Brillouin zone**.

## B. Matrix Optics of 1D Periodic Media

Each unit cell is a **lossless reciprocal** system, and represented by a generic unimodular **wave-transfer matrix**,  
And, the medium is a Bragg grating with an infinite number of segments (**unit cells**).



$$\begin{bmatrix} U_{m+1}^{(+)} \\ U_{m+1}^{(-)} \end{bmatrix} = \mathbf{M}_o \begin{bmatrix} U_m^{(+)} \\ U_m^{(-)} \end{bmatrix} \quad \text{: recurrence relations}$$

$$\mathbf{M}_o = \begin{bmatrix} 1/t^* & r/t \\ r^*/t^* & 1/t \end{bmatrix}$$

### *Eigenvalue Problem and Bloch Modes for the 1D periodic medium:*

By definition, the modes of the periodic medium are **self-reproducing** waves after transmission through a **distance  $\Lambda$** ,

$$\begin{bmatrix} U_{m+1}^{(+)} \\ U_{m+1}^{(-)} \end{bmatrix} = e^{-j\Phi} \begin{bmatrix} U_m^{(+)} \\ U_m^{(-)} \end{bmatrix}, \quad m = 1, 2, \dots; \quad \Phi = K\Lambda \quad \text{Bloch Phase}$$

For  $m = 0$ ,  $\mathbf{M}_o \begin{bmatrix} U_0^{(+)} \\ U_0^{(-)} \end{bmatrix} = e^{-j\Phi} \begin{bmatrix} U_0^{(+)} \\ U_0^{(-)} \end{bmatrix} \Rightarrow \mathbf{M}_o - e^{-j\Phi} \mathbf{I} = 0$  : Eigen value problem

Eigen values  $\rightarrow e^{-j\Phi} = \frac{1}{2}(1/t + 1/t^*) \pm j\{1 - [\frac{1}{2}(1/t + 1/t^*)]^2\}^{1/2} \Rightarrow \cos \Phi = \text{Re} \left\{ \frac{1}{t} \right\} \quad \Phi = K\Lambda$

Eigenvectors  $\rightarrow \begin{bmatrix} U_0^{(+)} \\ U_0^{(-)} \end{bmatrix} \propto \begin{bmatrix} r/t \\ e^{-j\Phi} - 1/t^* \end{bmatrix}$

**Example)** If the initial layer in the unit cell is a homogeneous medium of refractive index  $n_1$  and width  $d_1$ ,  
then The periodic function  $P_K(z)$  associated with the Bloch wave at distance  $z$  into this layer is

$$p_K(z) e^{-jKz} = U_0^{(+)} e^{-jn_1 k_0 z} + U_0^{(-)} e^{jn_1 k_0 z}, \quad 0 < z < d_1$$

$\Rightarrow p_K(z) \propto [-r e^{-jn_1 k_0 z} + (e^{-jK\Lambda} - 1) e^{jn_1 k_0 z}] e^{jKz}, \quad 0 < z < d_1$

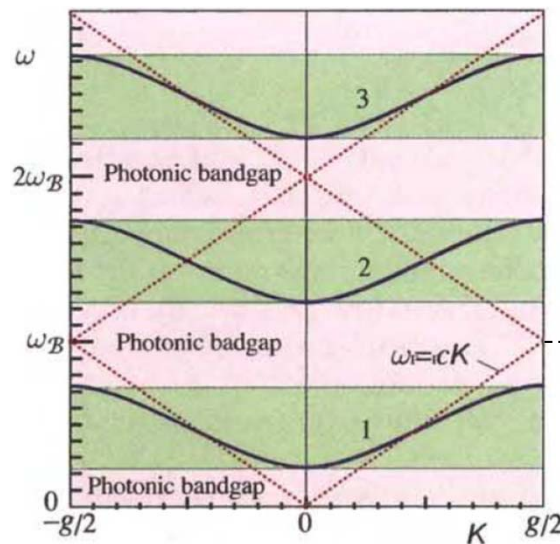
## Dispersion Relation and Photonic Band Structure for the 1D periodic medium:

The dispersion **relation** relates the Bloch wavenumber  $K$  and the angular frequency  $\omega$ .

$$\cos \Phi = \operatorname{Re} \left\{ \frac{1}{t} \right\} \xrightarrow[t = t(\omega)]{\Phi = K\Lambda} \cos \left( 2\pi \frac{K}{g} \right) = \operatorname{Re} \left\{ \frac{1}{t(\omega)} \right\} \quad g = 2\pi/\Lambda$$

→ Dispersion relation ( $\omega$ - $K$  relation)

- $\cos(2\pi K/g)$  is a periodic function of  $K$  of period  $g = 2\pi/\Lambda$
- It is common to limit the domain of the dispersion relation to a period with values of  $K$  in the interval  $[-g/2, g/2]$  or  $[-\pi/\Lambda, \pi/\Lambda]$  → (First Brillouin zone)
- Corresponding precisely to limiting the phase  $\Phi$  to  $[-\pi, \pi]$  → (First Brillouin zone)



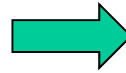
← Propagation regime, where  $|\cos \Phi| = |\operatorname{Re}\{1/t(\omega)\}| \leq 1$

← Photonic bandgap regime, where  $|\operatorname{Re}\{1/t(\omega)\}| > 1$

→ Bragg frequency  
where  $c = c_o/\bar{n}$  and  $\bar{n}$  is the average

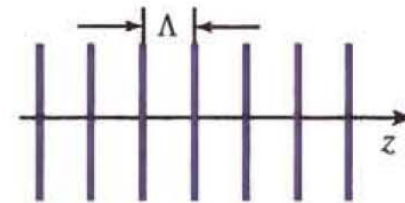
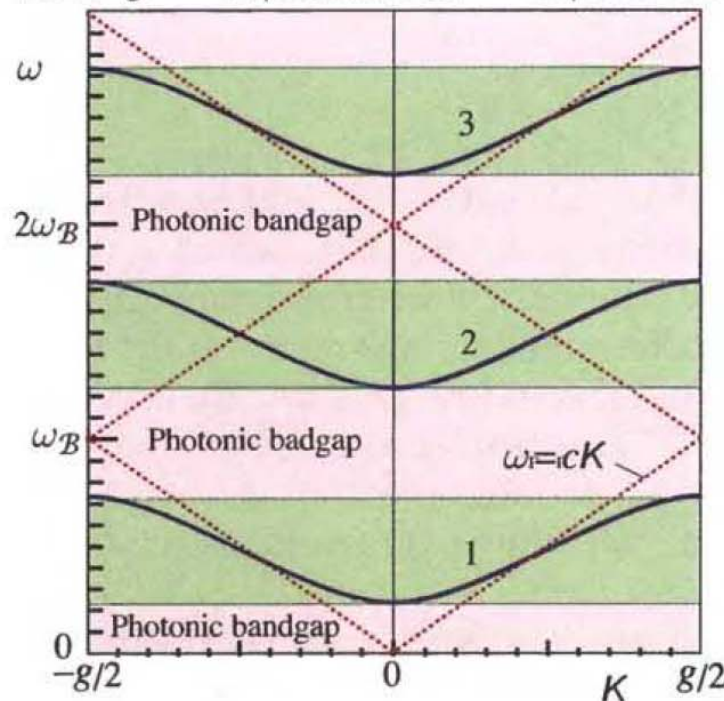
**EXAMPLE 7.2-1. Periodic Stack of Partially Reflective Mirrors.** The dispersion relation for a wave traveling along the axis of a periodic stack of identical partially reflective lossless mirrors with intensity reflectance  $|r|^2$  and intensity transmittance  $|t|^2 = 1 - |r|^2$ , separated by a distance  $\Lambda$ , is determined directly from Example 7.1-8. Using the results obtained there, namely  $\underline{t} = |t|e^{j\varphi}$  with  $\underline{\varphi} = nk_o\Lambda = (\omega/c)\Lambda$ , in conjunction with (7.2-13), provides the dispersion relation

$$\cos\left(2\pi\frac{K}{g}\right) = \operatorname{Re}\left\{\frac{1}{\underline{t}(\omega)}\right\}$$



$$\cos\left(2\pi\frac{K}{g}\right) = \frac{1}{|t|} \cos\left(\pi\frac{\omega}{\omega_B}\right), \quad (7.2-18)$$

where  $g = 2\pi/\Lambda$ , and  $\omega_B = c\pi/\Lambda$  is the Bragg frequency. This result is plotted in Fig. 7.2-4.



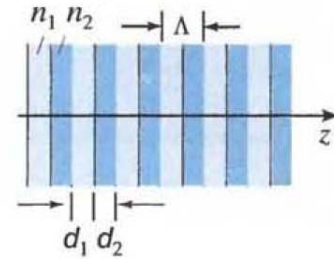
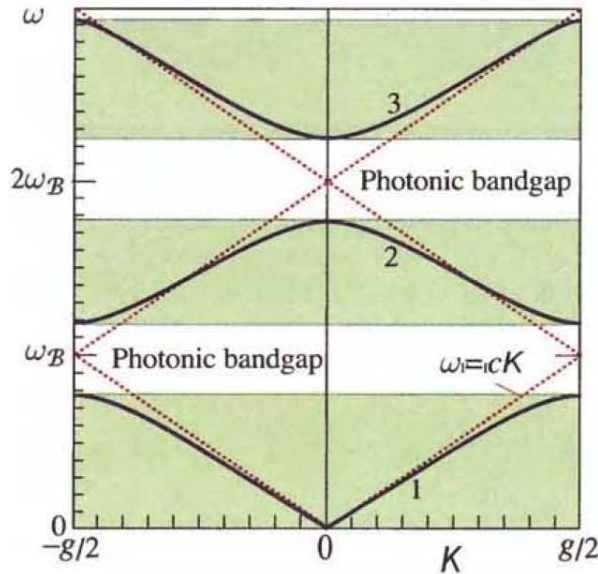
**Figure 7.2-4** Dispersion diagram of a periodic set of mirrors, each with intensity transmittance  $|t|^2 = 0.5$ , separated by a distance  $\Lambda$ . Here,  $\omega_B = \pi c/\Lambda$  and  $g = 2\pi/\Lambda$ . The dotted straight lines represent propagation in a homogeneous medium for which  $\omega/K = \omega_B/(g/2) = c$ .

**EXAMPLE 7.2-2. Alternating Dielectric Layers.** A periodic medium comprises alternating dielectric layers of refractive indexes  $n_1$  and  $n_2$ , with corresponding widths  $d_1$  and  $d_2$ , and period  $\Lambda = d_1 + d_2$ . This system is the dielectric Bragg grating described in Example 7.1-9 with  $N = \infty$ . For a wave traveling along the axis of periodicity,  $\text{Re}\{1/t\} = \text{Re}\{A\}$  is given by (7.1-61). Using the relations  $\varphi_1 + \varphi_2 = k_o(n_1 d_1 + n_2 d_2) = \pi\omega/\omega_B$  and  $\varphi_1 - \varphi_2 = \zeta\pi\omega/\omega_B$ , where  $\omega_B = (c_o/\bar{n})(\pi/\Lambda)$  is the Bragg frequency,  $\bar{n} = (n_1 d_1 + n_2 d_2)/\Lambda$  is the average refractive index and  $\zeta = (n_1 d_1 - n_2 d_2)/(n_1 d_1 + n_2 d_2)$ , (7.2-13) provides the dispersion relation

$$\cos\left(2\pi \frac{K}{g}\right) = \frac{1}{t_{12}t_{21}} \left[ \cos\left(\pi \frac{\omega}{\omega_B}\right) - |r_{12}|^2 \cos\left(\pi \zeta \frac{\omega}{\omega_B}\right) \right], \quad (7.2-19)$$

where  $t_{12}t_{21} = 4n_1 n_2 / (n_1 + n_2)^2$  and  $|r_{12}|^2 = (n_2 - n_1)^2 / (n_1 + n_2)^2$ .

An example of this dispersion relation is plotted in Fig. 7.2-5 for dielectric materials with  $n_1 = 1.5$  and  $n_2 = 3.5$ , and  $d_1 = d_2$ . As with the periodic stack of partially reflective mirrors considered in Example 7.2-1, the photonic bandgaps are centered at the frequencies  $\omega_B$  and its multiples, and occur at either the center of the Brillouin zone ( $K = 0$ ) or at its edge ( $K = g/2$ ). In this case, however, the frequency region surrounding  $\omega = 0$  admits propagating modes instead of a forbidden gap. Dielectric materials with lower contrast have bandgaps of smaller width, but the bandgaps exist no matter how small the contrast.



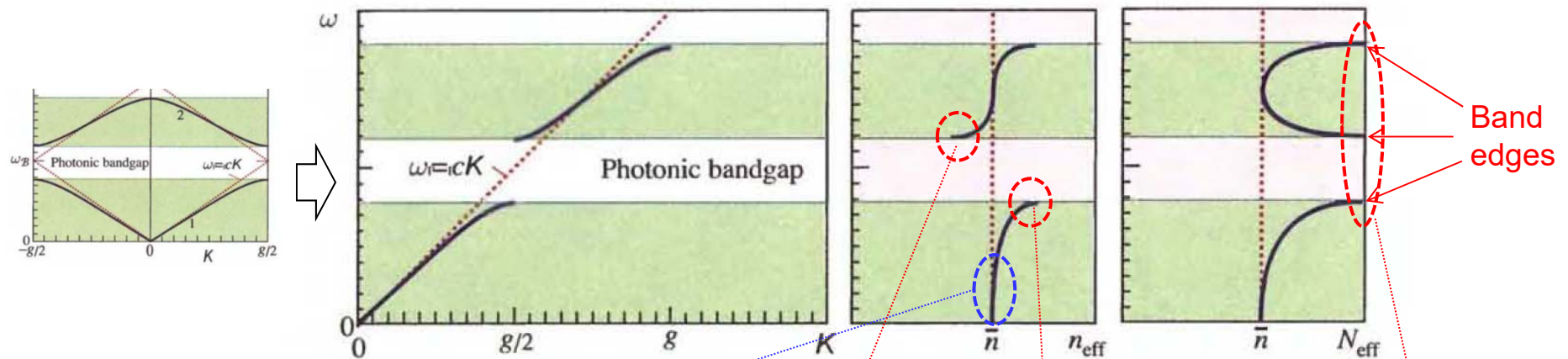
**Figure 7.2-5** Dispersion diagram of an alternating-layer periodic dielectric medium with  $n_1 = 1.5$  and  $n_2 = 3.5$ , and  $d_1 = d_2$ . Here,  $\omega_B = \pi c_o / \Lambda \bar{n}$  and  $g = 2\pi / \Lambda$ . The dotted straight lines represent propagation in a homogeneous medium of mean refractive index  $\bar{n}$ , so that  $\omega/K = \omega_B / (g/2) = c_o / \bar{n} = c$ .

## Phase and Group Velocities

phase velocity  $\omega/K$ , effective refractive index  $n_{\text{eff}} = c_0 K / \omega$

group velocity  $v = d\omega/dK$ , effective group index  $N_{\text{eff}} = c_0 dK/d\omega$

For alternating-layer dielectric periodic medium :



**Figure 7.2-6** Frequency dependence of the effective refractive index  $n_{\text{eff}}$ , which determines the phase velocity, and the effective group index  $N_{\text{eff}}$ , which determines the group velocity.

At low frequencies (long wavelengths) the material behaves as a homogeneous medium with the average refractive index ( $n_{\text{eff}} \sim \bar{n}$ ).

For the mode at the bottom of the upper band, greater energy is localized in the layers with the lower refractive index.

The mode at the top of the lower band, has greater energy in the dielectric layers with the higher refractive index.

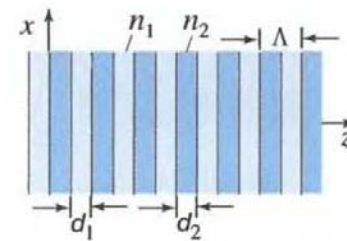
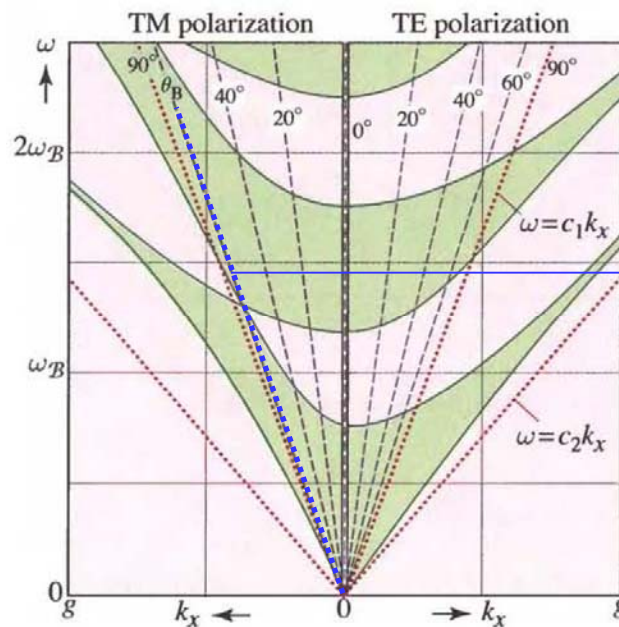
As the edges of the bandgap are approached, from below or above, this index increases substantially, so that the group velocity is much smaller, i.e., optical pulses are very slow near the edges of the bandgap.

## Off-Axis Dispersion Relation and Band Structure

$\cos(K\Lambda) = \text{Re}\{1/t(\omega)\}$  : The same equation to the case of on-axis, but now  $\text{Re}\{1/t(\omega)\}$  depends on the angles of incidence within the layers of each segment and on the state of polarization (TE or TM).

$$\text{Re}\left\{\frac{1}{t}\right\} = \frac{(\tilde{n}_1 + \tilde{n}_2)^2}{4\tilde{n}_1\tilde{n}_2} \cos(\tilde{\varphi}_1 + \tilde{\varphi}_2) - \frac{(\tilde{n}_2 - \tilde{n}_1)^2}{4\tilde{n}_1\tilde{n}_2} \cos(\tilde{\varphi}_1 - \tilde{\varphi}_2), \quad (7.1-62)$$

where  $\tilde{\varphi}_1 = n_1 k_o d_1 \cos \theta_1$  and  $\tilde{\varphi}_2 = n_2 k_o d_2 \cos \theta_2$ ;  $\tilde{n}_1 = n_1 \cos \theta_1$  and  $\tilde{n}_2 = n_2 \cos \theta_2$  for TE polarization; and  $\tilde{n}_1 = n_1 \sec \theta_1$  and  $\tilde{n}_2 = n_2 \sec \theta_2$  for TM polarization.



**Figure 7.2-7** Projected dispersion diagram for an alternating-layer periodic dielectric medium with  $n_1 = 1.5$ ,  $n_2 = 3.5$ , and  $d_1 = d_2 = \Lambda/2$ . Here,  $\omega_B = \pi c_o / \Lambda \bar{n}$  and  $g = 2\pi / \Lambda$ . Photonic bands are shaded (green). The dashed lines represent fixed angles of incidence  $\theta_1$  in layer 1, including the Brewster angle  $\theta_B = 66.8^\circ$ . Points within the region bounded by the light lines  $\omega = c_1 k_x$  and  $\omega = c_2 k_x$  represent normal-to-axis waves.

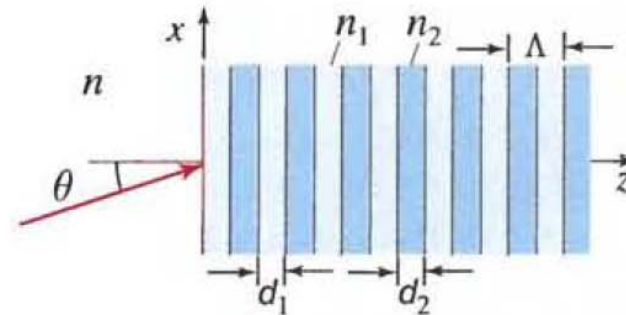
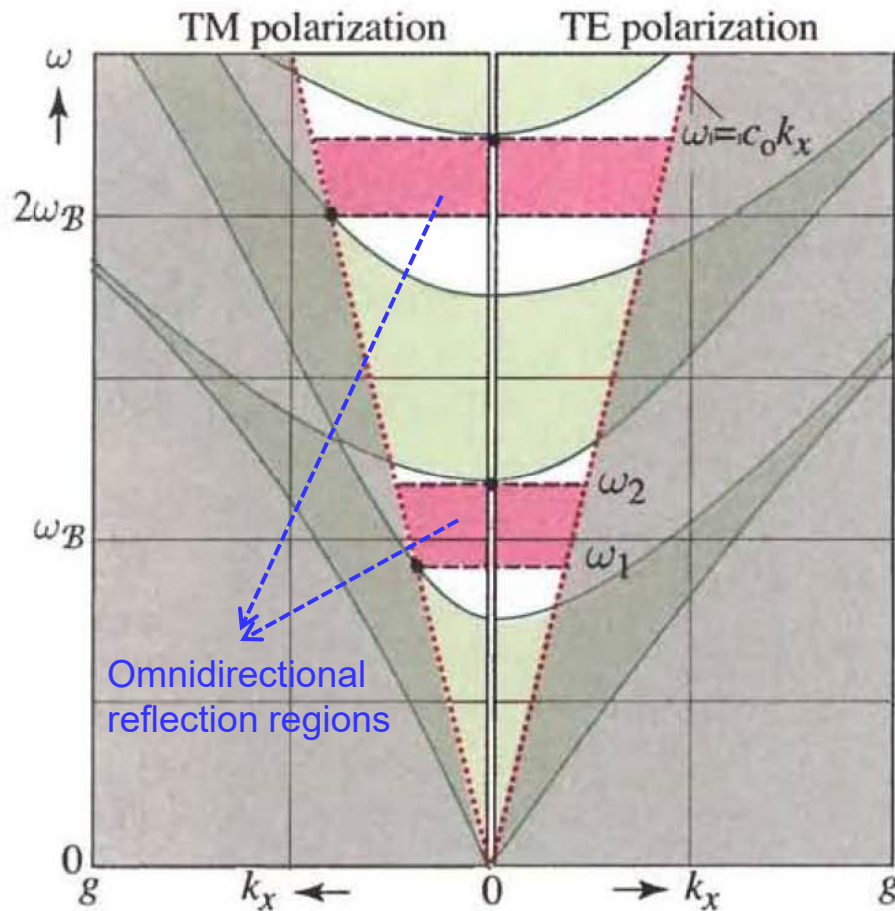
Oblique TM wave propagating at the Brewster angle,  $\theta_B = \tan^{-1}(n_2/n_1)$  photonic bandgap cannot occur at all.

Whether there exists a frequency range over which propagation is forbidden at all angles of incidence  $\theta_1$  and  $\theta_2$  and for both polarizations (Complete Photonic Bandgap) ?

➔ Complete photonic bandgaps cannot exist within 1 D periodic structures!! ➔ 2D or 3D structure is needed.

## Omnidirectional Reflection

Under certain conditions and within a specified range of angular frequencies, the periodic medium acts as a perfect mirror, totally reflecting waves incident from any direction and with any polarization!



**Figure 7.2-10** Projected dispersion diagram for an alternating-layer dielectric medium with  $n_1 = 1.5$ ,  $n_2 = 3.5$ , and  $d_1 = d_2 = \Lambda/2$ . The dotted lines (red) are light lines for a homogeneous medium with refractive index  $n = 1$ . In the spectral band between  $\omega_1$  and  $\omega_2$ , the medium acts as a perfect omnidirectional reflector for all polarizations. A similar band is shown at higher angular frequencies.

The red dashed line the  $\omega - k_x$  region that can be accessed by waves entering from the homogeneous

This region is thus bounded by the line  $\omega = (c_0/n)k_x$  ( $k_x = (\omega/c_0)n \sin \theta$ ) ( $n = 1$  in this figure)