4. Fresnel and Fraunhofer diffraction

Augustin Jean Fresnel (1788-1827)
Made contributions to transverse nature of light and diffraction theory

Josef von Fraunhofer (1787-1826)
Developed diffraction gratings and increased understanding of diffraction

Remind! Diffraction under paraxial approx.

$$r = \sqrt{z_i^2 + (x_i - x_0)^2 + (y_i - y_0)^2}$$

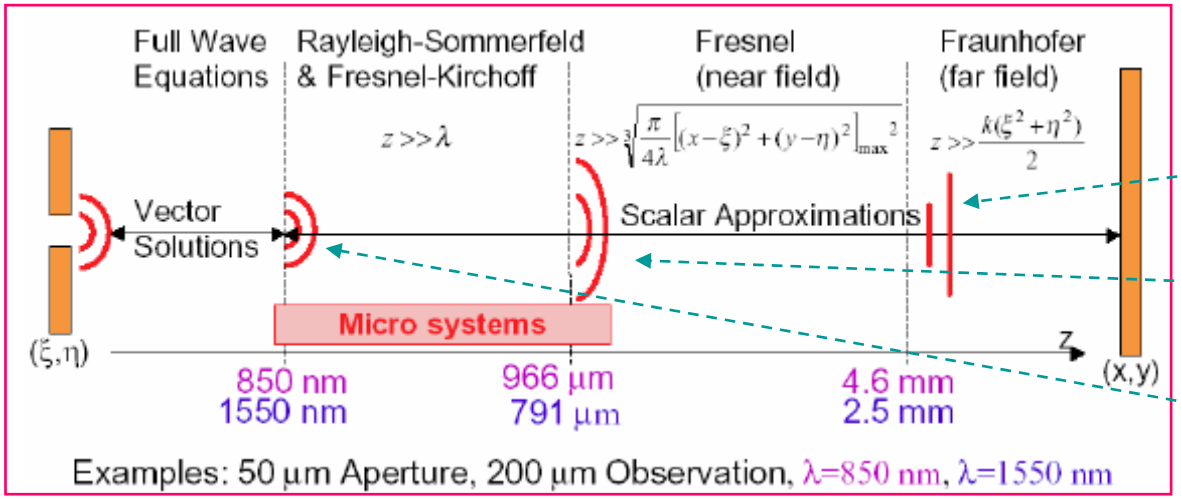
(paraxial) $z_i^2 \gg (x_i - x_0)^2 + (y_i - y_0)^2 \longrightarrow r \doteqdot z_i \left(1 + \frac{(x_i - x_0)^2 + (y_i - y_0)^2}{2z_i^2}\right)$

Fresnel region (near-field region)

$$r = z_i + \frac{x_i^2 + y_i^2}{2z_i} - \frac{x_i x_o + y_i y_o}{z_i} + \frac{x_o^2 + y_o^2}{2z_i}$$

Fraunhofer region (far-field region)

$$r = z_i + \frac{x_i^2 + y_i^2}{2z_i} - \frac{x_i x_o + y_i y_o}{z_i}$$



Fraunhofer Approximation
- Assume planar wavefronts

Fresnel Approximation
- Assume parabolic wavefronts

Rayleigh-Sommerfeld Formulation
- Assume Spherical wavefronts

Remind! Huygens-Fresnel principle

"Every unobstructed point of a wavefront, at a given instant in time, serves as a source of secondary wavelets (with the same frequency as that of the primary wave).
The amplitude of the optical field at any point beyond is the superposition of all these wavelets (considering their amplitude and relative phase)."

Huygens's principle:
By itself, it is unable to account for the details of the diffraction process.
It is indeed independent of any wavelength consideration.

Fresnel's addition of the concept of interference

$$\psi(P) = A\,\frac{\exp\left[-i(\omega t - kr_0)\right]}{r_0}\iint_S \frac{\exp(iks)}{s}K(\chi)\,dS$$

Remind! After the Huygens-Fresnel principle

Fresnel's shortcomings :

He did not mention the existence of backward secondary wavelets, however, there also would be a reverse wave traveling back toward the source. He introduce a quantity of the obliquity factor, but he did little more than conjecture about this kind.

$$\psi(P) = A\frac{\exp\left[-i(\omega t - kr_0)\right]}{r_0}\iint_S \frac{\exp(iks)}{s}K(\chi)\,dS$$

Gustav Kirchhoff : Fresnel-Kirhhoff diffraction theory

A more rigorous theory based directly on the solution of the differential wave equation. He, although a contemporary of Maxwell, employed the older elastic-solid theory of light. He found $K(\chi) = (1 + \cos\theta)/2$. $K(0) = 1$ in the forward direction, $K(\pi) = 0$ with the back wave.

$$\psi(P) = -\left\{\frac{ia}{2\lambda}\right)\iint_A \left[\frac{\exp(ikr)}{r}\right]\left[\frac{\exp(iks)}{s}\right][\cos(n,r) - \cos(n,s)]\,dS$$

Arnold Johannes Wilhelm Sommerfeld : Rayleigh-Sommerfeld diffraction theory

A very rigorous solution of partial differential wave equation. The first solution utilizing the electromagnetic theory of light.

$$\psi(P) = -\left(\frac{ia}{\lambda}\right)\iint_A \left[\frac{\exp(ikr)}{r}\right]\left[\frac{\exp(iks)}{s}\right]\cos(n,s)\,dS$$

Remind! Huygens-Fresnel Principle revised by 1st R – S solution

$$U(P_0) = \frac{1}{j\lambda} \iint_{\Sigma} U(P_1) \frac{\exp(jkr_{01})}{r_{01}} \cos\theta \, ds$$

H-F principle

$$amplitude \quad \propto \quad \frac{1}{\lambda} \propto \nu$$

$$phase: \quad \frac{1}{j} \text{ (lead the incident phase by } 90^{\circ}\text{)}$$

$$U(P_0) = \iint_{\Sigma} h(P_0, P_1) U(P_1) ds$$

$$h(P_0, P_1) = \frac{1}{j\lambda} \frac{\exp(jkr_{01})}{r_{01}} \cos\theta$$

Impulse response function

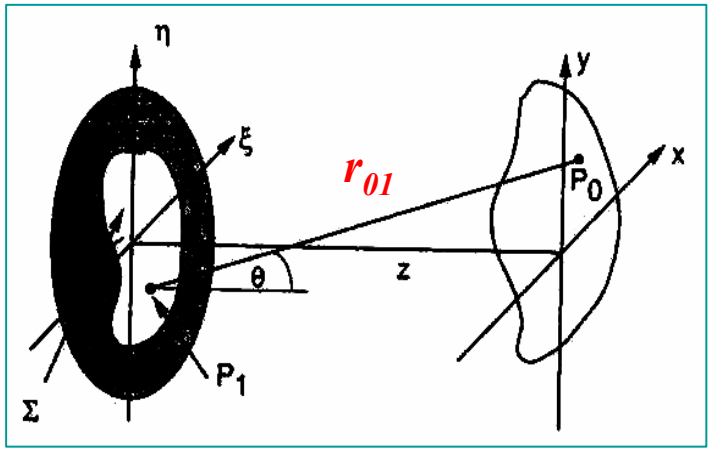
Each secondary wavelet has a directivity pattern *cos*θ

Huygens-Fresnel Principle

$$\cos\theta = \frac{z}{r_{01}}$$

$$U(P_0) = \frac{1}{j\lambda}\iint_{\Sigma} U(P_1)\frac{\exp(jkr_{01})}{r_{01}}\cos\theta\, ds$$

$$U(x,y) = \frac{z}{j\lambda}\iint_{\Sigma} U(\xi,\eta)\frac{\exp(jkr_{01})}{r_{01}^2}\, d\xi\, d\eta$$



Aperture (ξ,η) Screen (x,y)

$$r_{01} = \sqrt{z^2 + (x-\xi)^2 + (y-\eta)^2}$$

Note! So far we have used only two assumptions (R-S solution) :

Scalar wave equation + ($r_{01} >> \lambda$)

Fresnel (paraxial) Approximation

$$r_{01} \approx z\left[1+\frac{1}{2}\left(\frac{x-\xi}{z}\right)^2+\frac{1}{2}\left(\frac{y-\eta}{z}\right)^2\right]$$

$$U(x,y)=\frac{e^{jkz}}{j\lambda z}\int\int_{-\infty}^{\infty}U(\xi,\eta)\exp\left\{j\frac{k}{2z}\left[(x-\xi)^2+(y-\eta)^2\right]\right\}d\xi d\eta$$

Since k is 10^7 order, the quadratic terms in r_{01} must be considered in the exponent.

$$U(x,y)=\frac{e^{jkz}}{j\lambda z}e^{j\frac{k}{2z}\left(x^2+y^2\right)}\int\int_{-\infty}^{\infty}\left\{U(\xi,\eta)e^{j\frac{k}{2z}\left(\xi^2+\eta^2\right)}\right\}e^{-j\frac{2\pi}{\lambda z}(x\xi+y\eta)}d\xi d\eta$$

$$\boxed{U(x,y)=\frac{e^{jkz}}{j\lambda z}e^{j\frac{k}{2z}\left(x^2+y^2\right)}\mathcal{F}\left\{U(\xi,\eta)e^{j\frac{k}{2z}\left(\xi^2+\eta^2\right)}\right\}\Bigg|_{f_X=x/\lambda z,\, f_Y=y/\lambda z}}$$

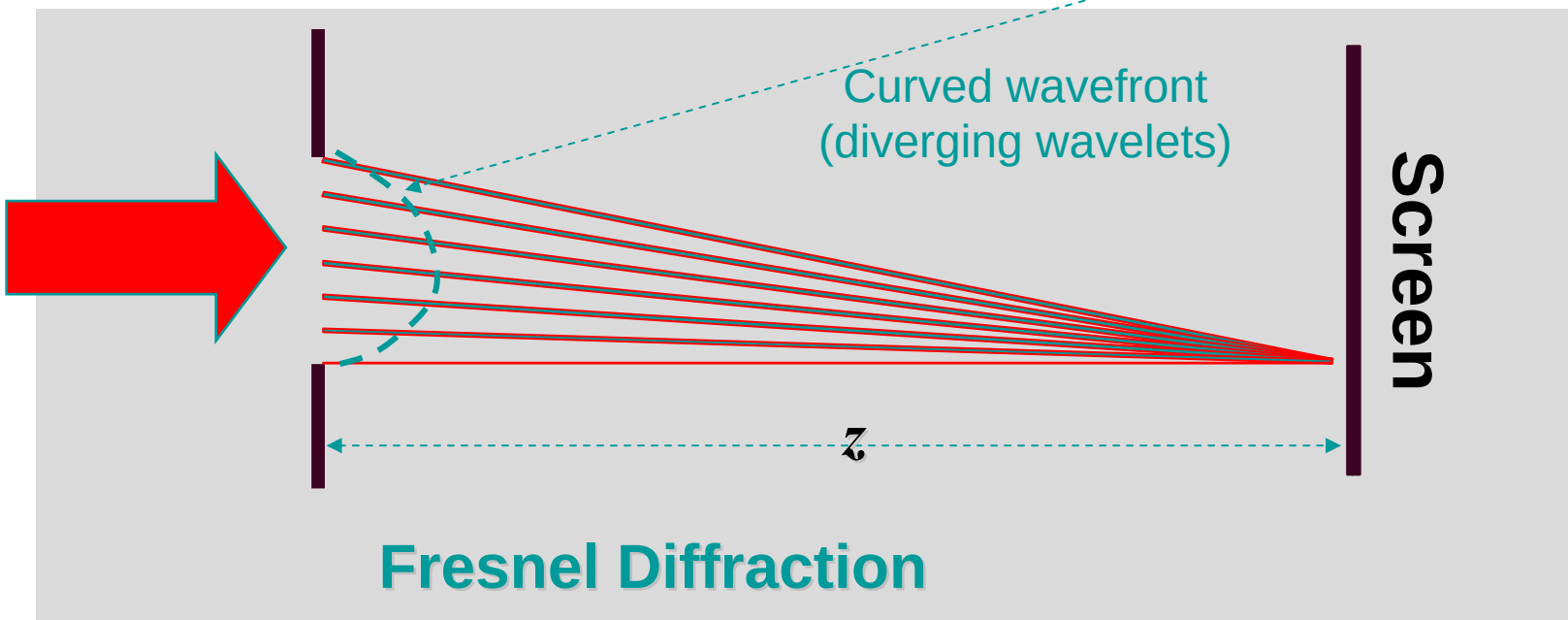
Fresnel diffraction integral

Fresnel (near-field) diffraction

This is most general form of diffraction

- **No restrictions on optical layout**
  - **near-field diffraction**
  - **curved wavefront**
- **Analysis somewhat difficult**

$$U(x,y) \approx \mathcal{F}\left\{U(\xi,\eta)\, e^{j\frac{k}{2z}\left(\xi^2+\eta^2\right)}\right\}$$

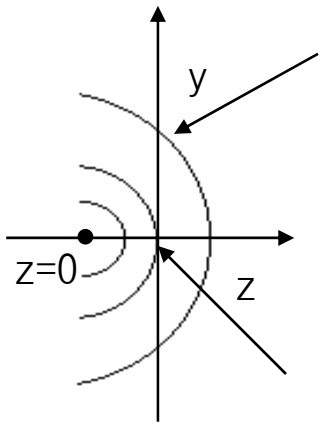


Positive phase and Negative phase

$$\exp(jkr_{01})$$

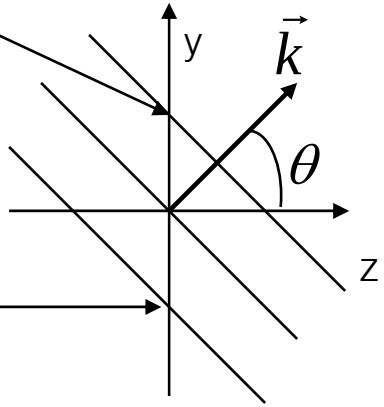
$$\exp\left[j\frac{k}{2z}\left(x^2+y^2\right)\right]$$

diverging



Wavefront emitted earlier

Wavefront emitted later

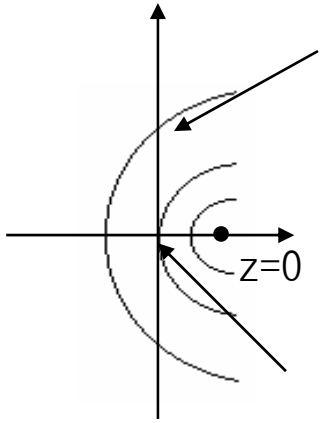


$$\exp(j2\pi\alpha y)$$

$$\exp(-jkr_{01})$$

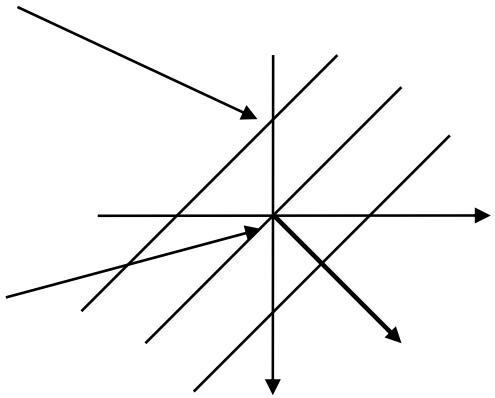
$$\exp\left[-j\frac{k}{2z}\left(x^2+y^2\right)\right]$$

converging



Wavefront emitted later

Wavefront emitted earlier



$$\exp(-j2\pi\alpha\, y)$$

Accuracy of the Fresnel Approximation

$$z^3 \gg \frac{\pi}{4\lambda}\left[(x-\xi)^2+(y-\eta)^2\right]^2_{\max}$$

- Accuracy can be expected for much shorter distances

for $U(\xi,\eta)$ *smooth & slow varing function;* $2|x-\xi| = D \le 4\sqrt{\lambda z}$

$$z \ge \frac{D^2}{16\lambda}$$ Fresnel approximation

Impulse Response and Transfer Functions of Fresnel diffraction

$$U(x,y)=\frac{e^{jkz}}{j\lambda z}\int\int_{-\infty}^{\infty}U(\xi,\eta)\exp\left\{j\frac{k}{2z}\left[(x-\xi)^2+(y-\eta)^2\right]\right\}d\xi d\eta$$

$$\Downarrow$$

$$U(x,y)=\int\int_{-\infty}^{\infty}U(\xi,\eta)\,h(x-\xi,y-\eta)\,d\xi d\eta$$

This convolution relation means that Fresnel diffraction is linear shift-invariant.

$$h(x,y)=\frac{e^{jkz}}{j\lambda z}\exp\left[\frac{jk}{2z}\left(x^2+y^2\right)\right]$$

Impulse response function of Fresnel diffraction

Impulse Response and Transfer Functions of Fresnel diffraction

$$H\left(f_X, f_Y\right) \equiv \mathcal{F}\left\{h(x,y)\right\}$$

$$= \mathcal{F}\left\{\frac{e^{jkz}}{j\lambda z}\exp\left[j\frac{\pi}{\lambda z}\left(x^2+y^2\right)\right]\right\} = e^{jkz}\exp\left[-j\pi\lambda z\left(f_X^2+f_Y^2\right)\right]$$

Remind! Transfer function of wave propagation phenomenon in free space

$$\varepsilon\left(f_X, f_Y; z\right) = \varepsilon\left(f_X, f_Y; 0\right)\exp\left[j2\pi\frac{z}{\lambda}\sqrt{1-\left(\lambda f_X\right)^2-\left(\lambda f_Y\right)^2}\right]$$

$$H\left(f_X, f_Y\right) = \exp\left[j2\pi\frac{z}{\lambda}\sqrt{1-\left(\lambda f_X\right)^2-\left(\lambda f_Y\right)^2}\right]$$

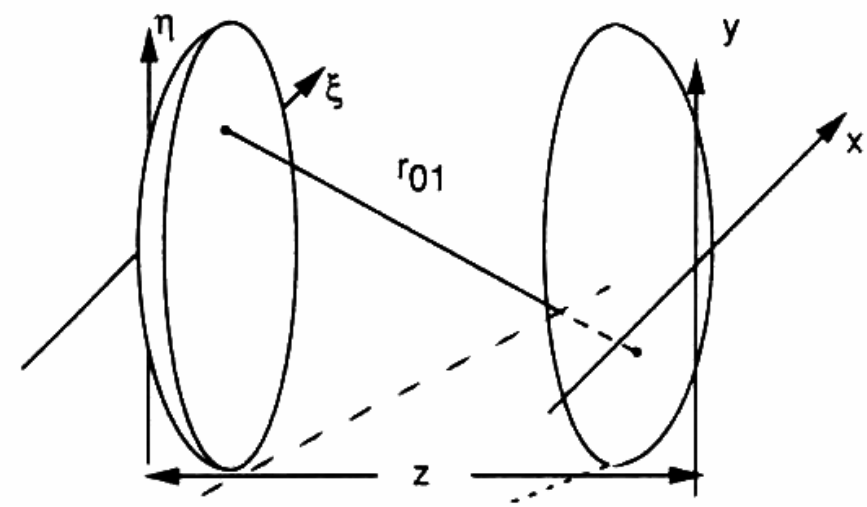
Under the paraxial approximation, the transfer function in free space becomes

$$\sqrt{1-\left(\lambda f_X\right)^2-\left(\lambda f_Y\right)^2} \approx 1-\frac{\left(\lambda f_X\right)^2}{2}-\frac{\left(\lambda f_Y\right)^2}{2}$$

$$H\left(f_X, f_Y\right) = e^{j\frac{2\pi}{\lambda}z}\exp\left[-j\pi\lambda z\left(f_X^2+f_Y^2\right)\right]$$

We confirm again that
the Fresnel diffraction
is paraxial approximation

Fresnel Diffraction between Confocal Spherical Surfaces



$$r_{01} = \sqrt{z^2 + (x-\xi)^2 + (y-\eta)^2}$$

$$r_{01} \approx z - \frac{x\xi}{z} - \frac{y\eta}{z}$$

$$U(x,y) = \frac{e^{jkz}}{j\lambda z} \int \int_{-\infty}^{\infty} U(\xi,\eta) e^{-j\frac{2\pi}{\lambda z}(x\xi + y\eta)} d\xi d\eta$$

$$= \frac{e^{jkz}}{j\lambda z} \mathcal{F}\{U(\xi,\eta)\}\Big|_{f_X = x/\lambda z,\, f_Y = y/\lambda z}$$

The Fresnel diffraction makes the exact Fourier transforms between the two spherical caps.

In summary, Fresnel diffraction is …

Assume: $z >> x_1, y_1; x_0, y_0$

$$\iint_{\Sigma} \rightarrow \iint_{-\infty}^{\infty} \qquad \text{(U = 0 outside the aperture)}$$

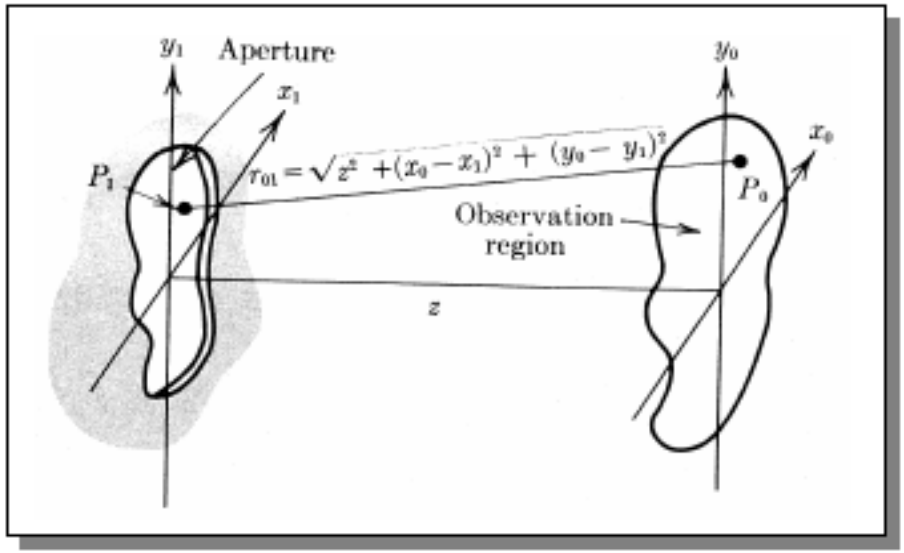
Fresnel's approximation:

In the exponent:

$$r = \sqrt{z^2 + (x_0 - x_1)^2 + (y_0 - y_1)^2}$$

$$\approx z\left[1 + \frac{1}{2}\left(\frac{x_0 - x_1}{z}\right)^2 + \frac{1}{2}\left(\frac{y_0 - y_1}{z}\right)^2\right] \qquad \left(\sqrt{1+x} \approx 1 + \frac{1}{2}x\right)$$

In the denominator: $r \rightarrow z$



$$U(x_0, y_0) = -\frac{ik}{2\pi}\frac{e^{ikz}}{z}\iint_{-\infty}^{\infty} U(x_1, y_1)\, e^{\frac{ik}{2z}\left[(x_0 - x_1)^2 + (y_0 - y_1)^2\right]}\, dx_1 dy_1$$

Fraunhofer Approximation

$$z \gg \frac{k\left(\xi^2+\eta^2\right)_{\max}}{2} \longrightarrow r_{01} \approx z\left[1-\frac{x\xi}{z}-\frac{y\eta}{z}\right]$$

$$U(x,y)=\frac{e^{jkz}e^{j\frac{k}{2z}(x^2+y^2)}}{j\lambda z}\int\int_{-\infty}^{\infty}U(\xi,\eta)\exp\left[-j\frac{2\pi}{\lambda z}(x\xi+y\eta)\right]d\xi d\eta$$

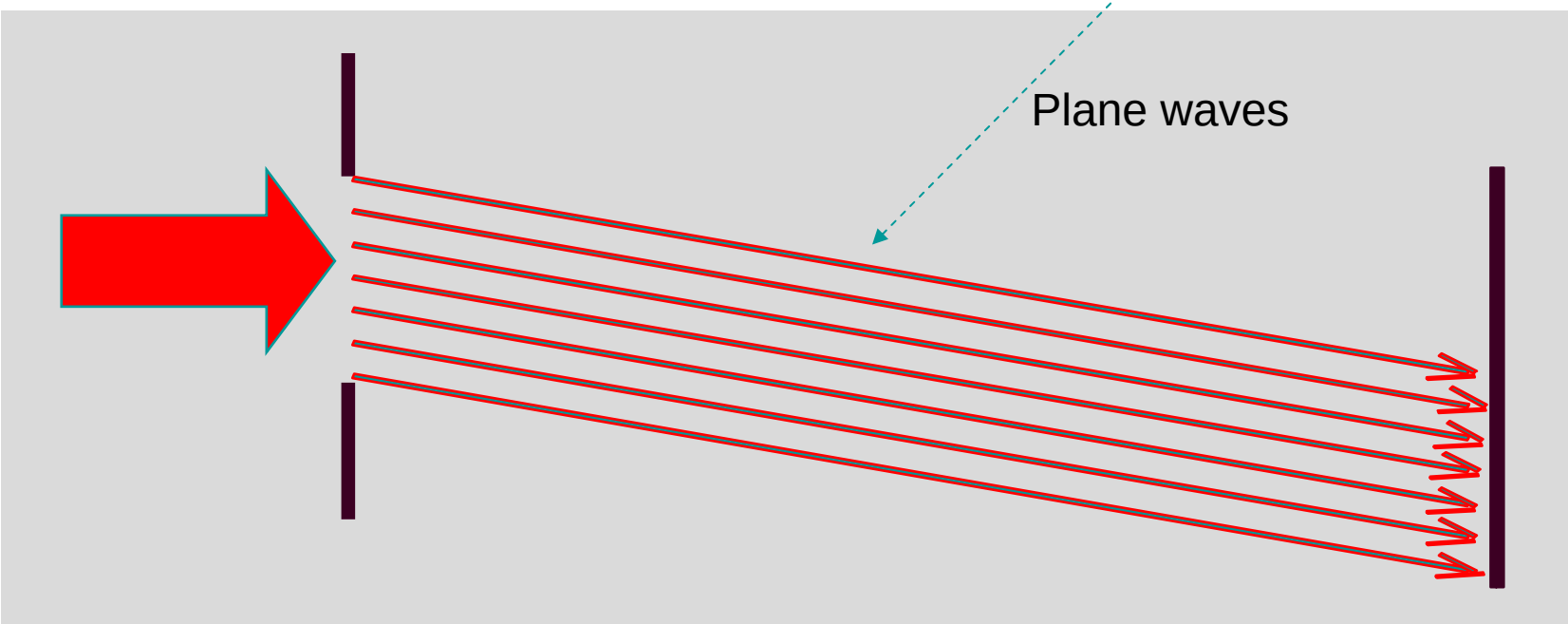
$$=\frac{e^{jkz}e^{j\frac{k}{2z}(x^2+y^2)}}{j\lambda z}\mathcal{F}\{U(\xi,\eta)\}\Big|_{f_X=x/\lambda z,\, f_Y=y/\lambda z}$$

$$z > \frac{2D^2}{\lambda} : \ \textit{anttena designer's formular}$$

Fraunhofer (far-field) diffraction

- **Specific sort of diffraction**
  - **far-field diffraction**
  - **plane wavefront**
  - **Simpler maths**

$$U(x,y) \approx \int\int_{-\infty}^{\infty} U(\xi,\eta)\exp\left[-j\frac{2\pi}{\lambda z}(x\xi + y\eta)\right]d\xi d\eta$$

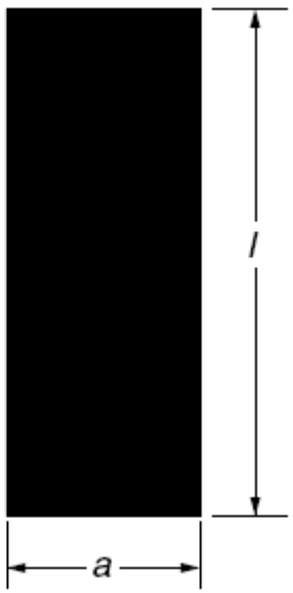


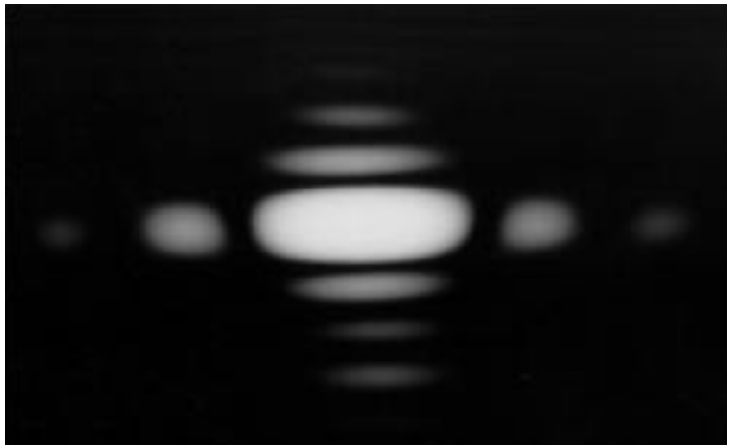
Fraunhofer diffraction pattern of a rectangular aperture with dimensions $a \times l$

The input function $$E(x_0, y_0) = \Pi\left(\frac{x_0}{a}\right)\Pi\left(\frac{y_0}{l}\right)$$

$E(x_i, y_i)$ in the $z = z_i$ plane

$$\boxed{E(x_i, y_i) = \frac{al}{j\lambda z_i} e^{jk[z_i + (x_i^2 + y_i^2)/2z_i]} \operatorname{sinc}\left(a\frac{x_i}{\lambda z_i}\right) \operatorname{sinc}\left(l\frac{y_i}{\lambda z_i}\right)}$$





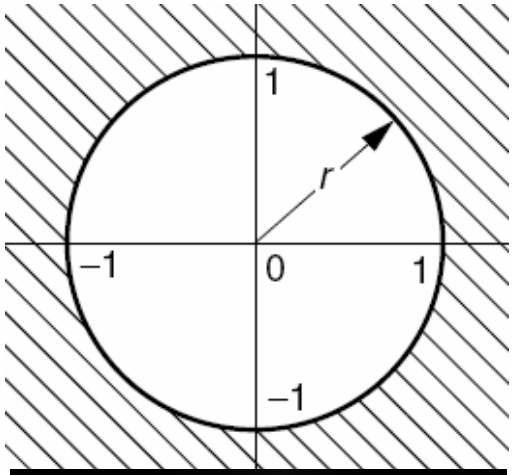
x and y can be transformed separately

Circular aperture $circ\left(\frac{r}{a}\right)=\begin{cases}1 & r\le a\\ 1 & r>a\end{cases}$

The Fourier transform of the circle function is $B\{\mathrm{circ}(r)\} = 2\pi \int_0^1 rJ_0(2\pi\rho r)\,dr$

Using $\int xJ_0(x)\,dx = xJ_1(x)$, and the similarity theorem; $B\{g(ar)\} = \frac{1}{a^2}G\left(\frac{\rho}{a}\right)$

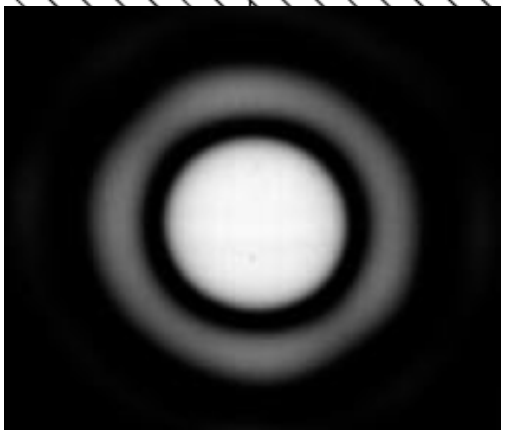
$$B\left\{\mathrm{circ}\left(\frac{r}{a}\right)\right\} = \frac{a}{\rho}J_1(2\pi\rho a)$$



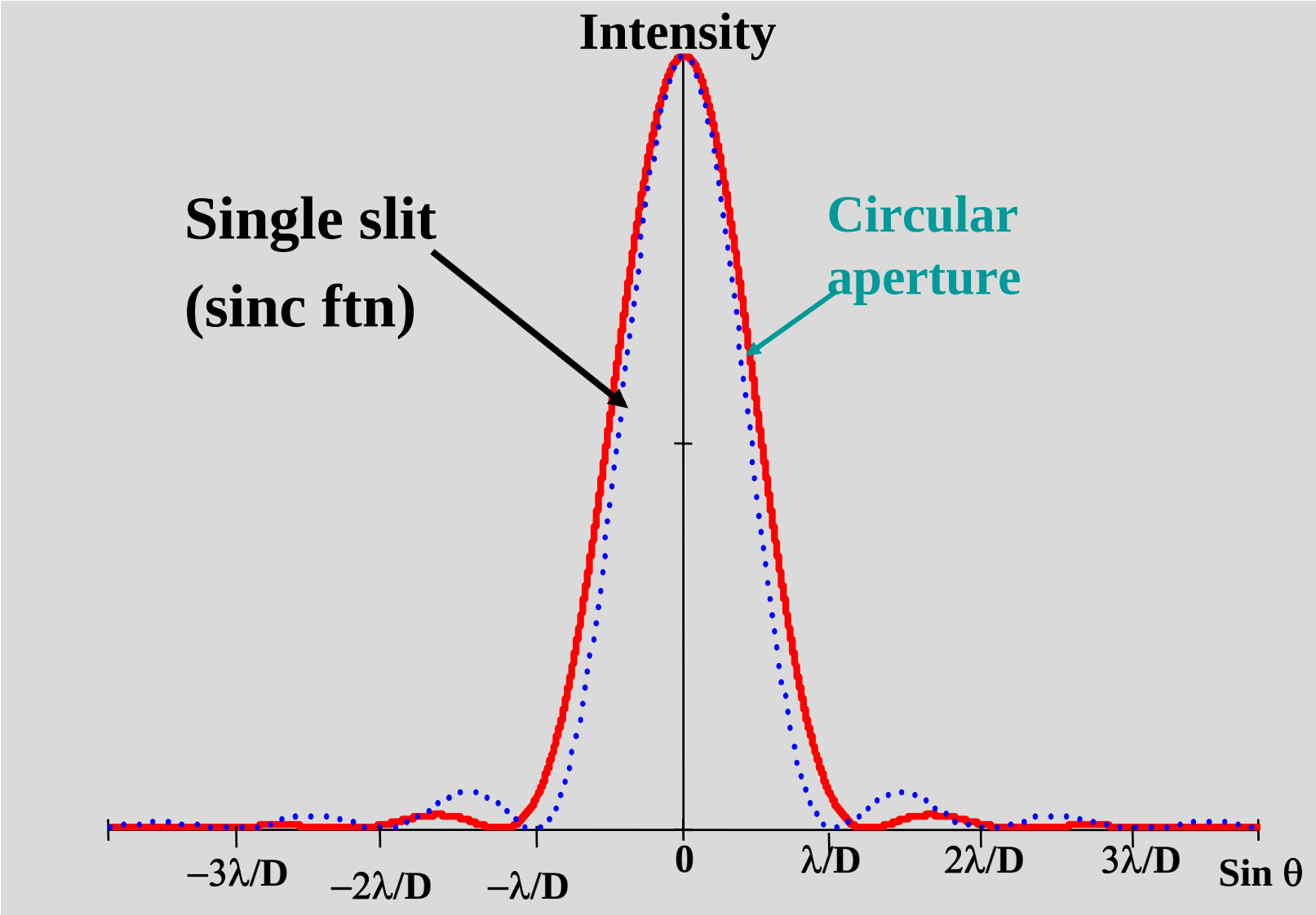
The intensity pattern of the diffraction

$$I(r_i) = \left(\frac{a^2}{\lambda z_i}\right)^2 \left(\frac{J_1(2\pi a\rho)}{a\rho}\right)^2 \quad \text{with } \rho = \frac{r_i}{\lambda z_i}$$

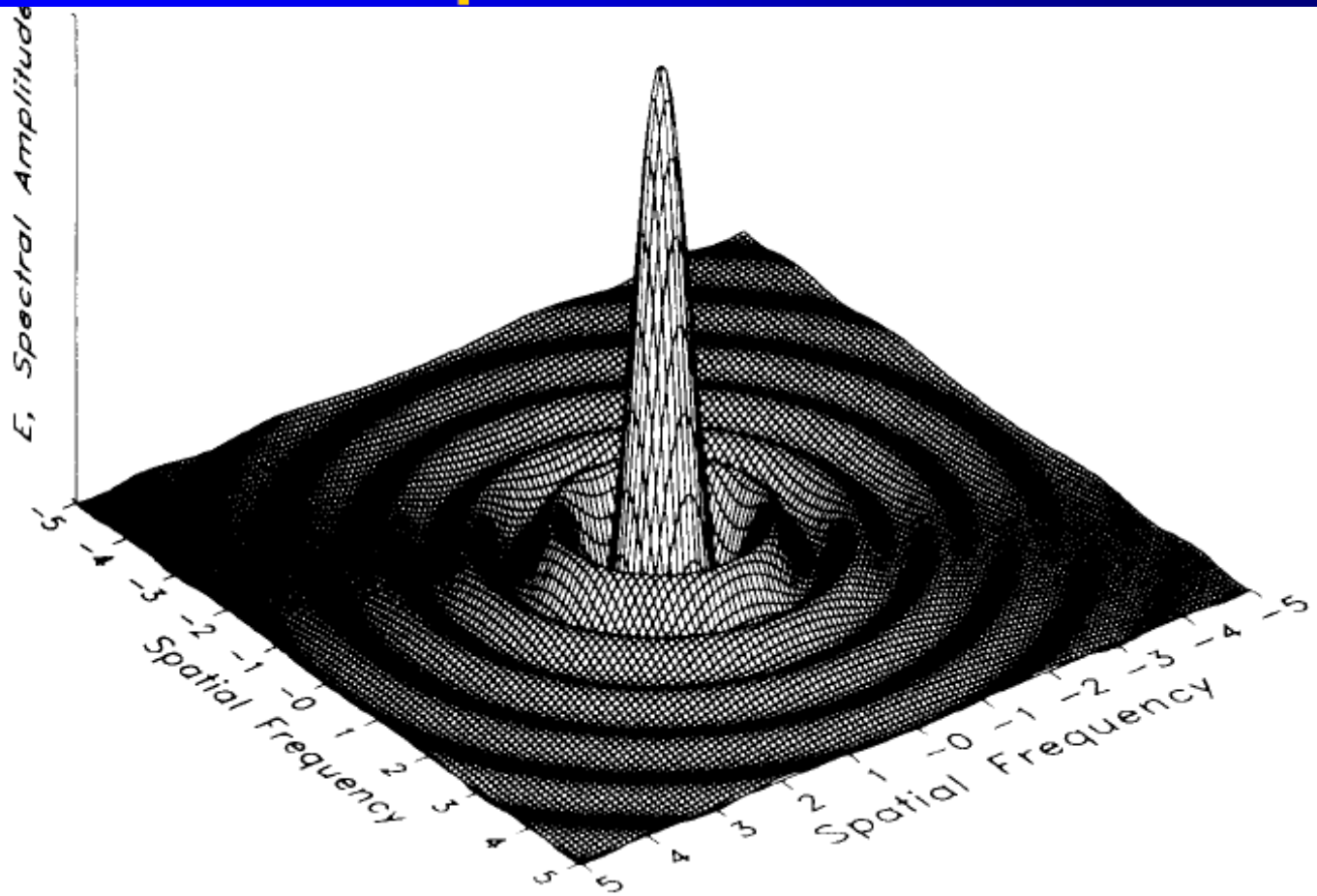
$$\boxed{I(r_i) = \left(\frac{ka^2}{z_i}\right)^2 \left(\frac{J_1(kar_i/z_i)}{kar_i/z_i}\right)^2}$$



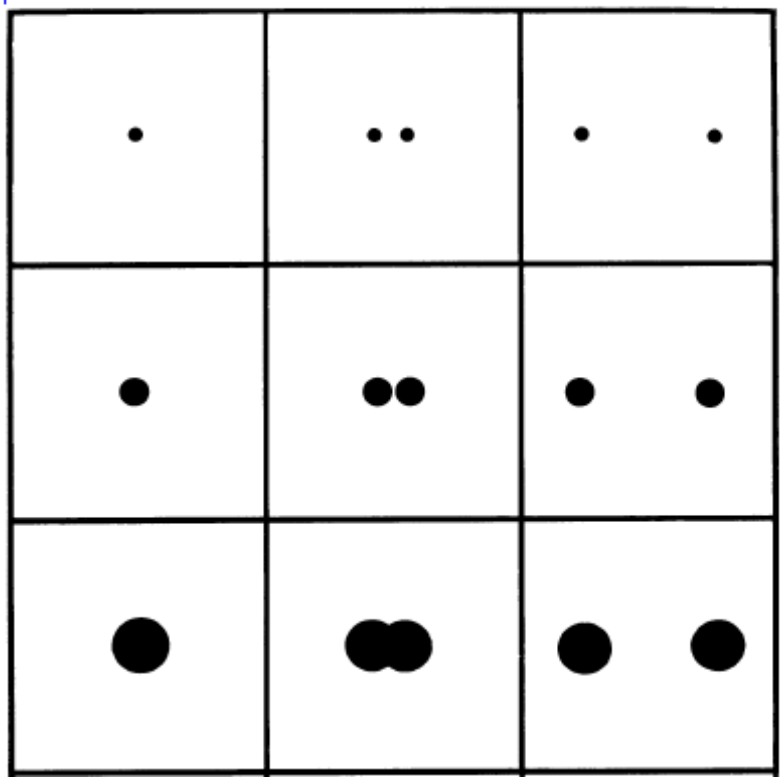
Circular Aperture

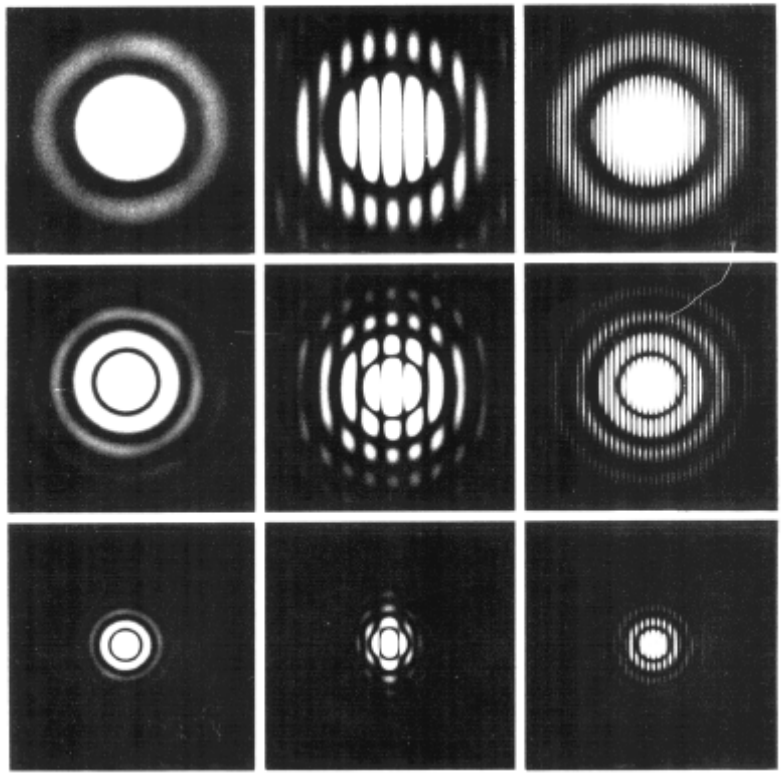


Circular Aperture Diffraction

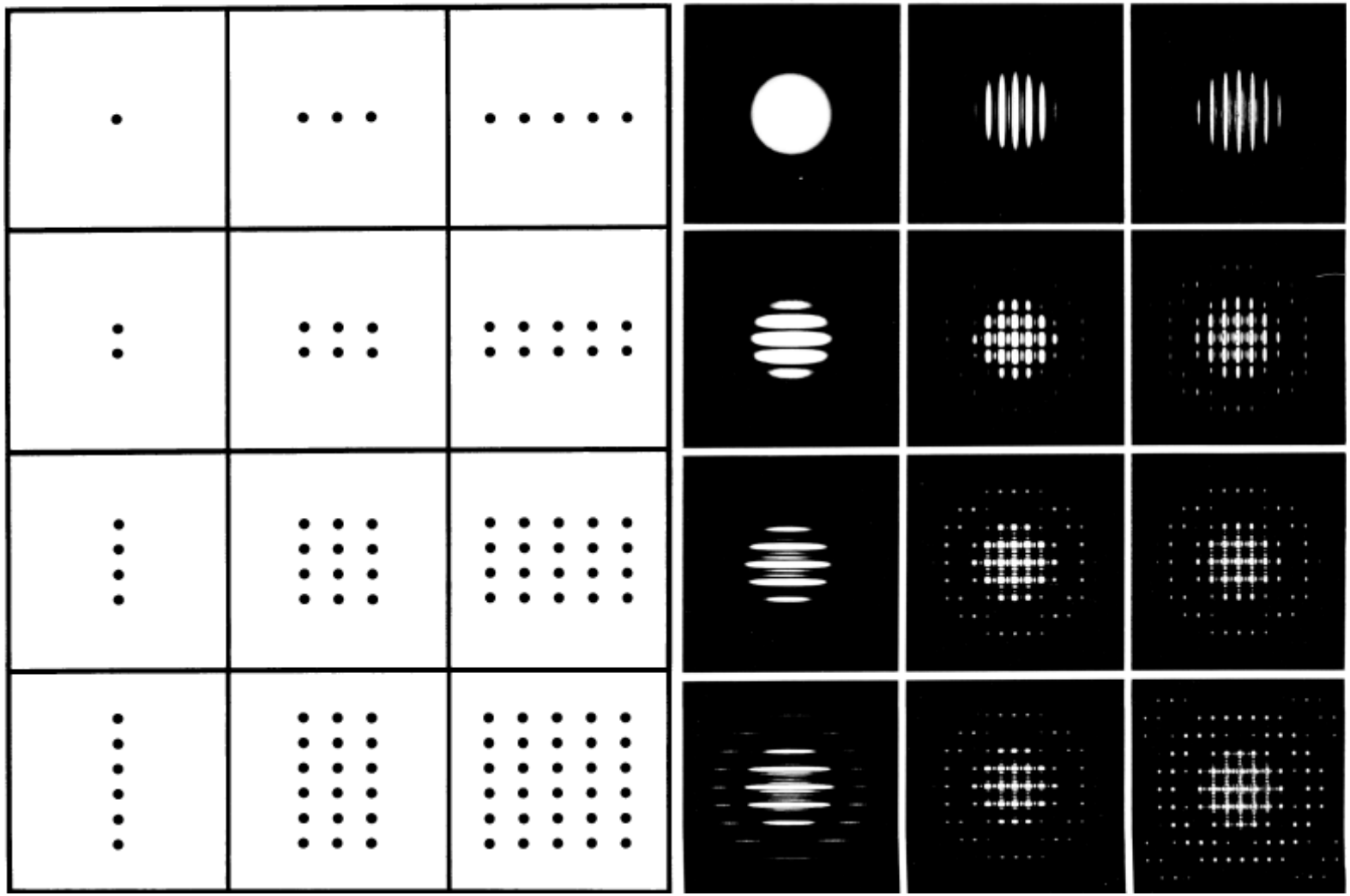


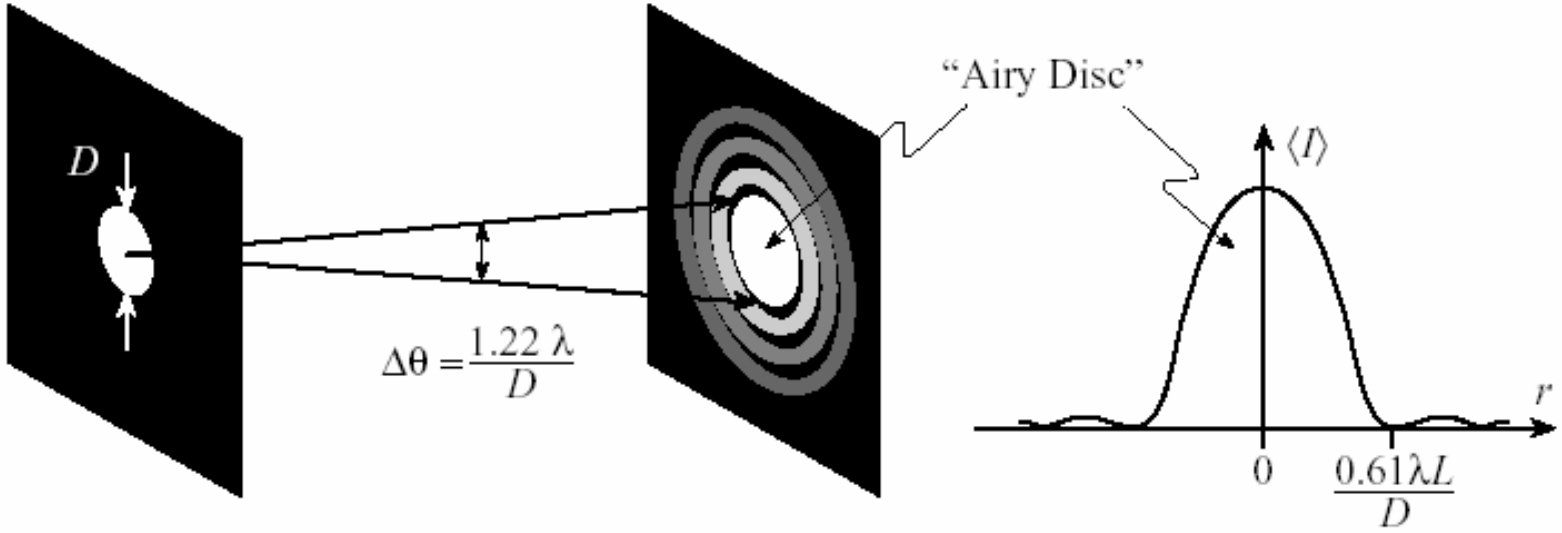
Scaling and Shifting for Circular Apertures

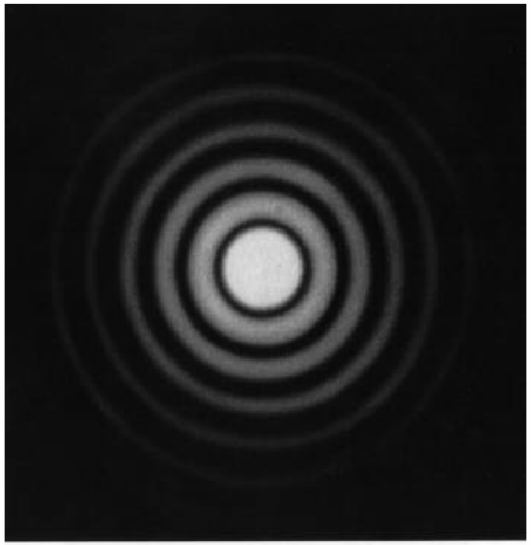




Multiple Circular Apertures



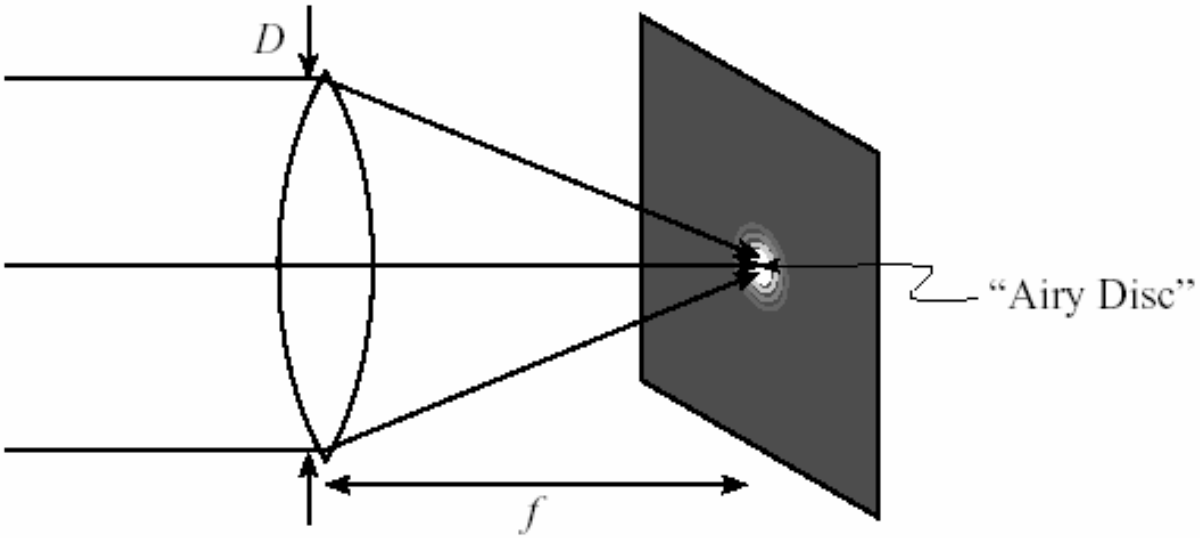

D
$\Delta\theta = \frac{1.22\,\lambda}{D}$
"Airy Disc"
$\langle I \rangle$
r
0
$\frac{0.61\lambda L}{D}$



Airy Disc

- Similar pattern to single slit
  - circularly symmetrical
- First zero point when $\frac{1}{2}kD \sin\theta = 3.83$
  - **$\sin\theta = 1.22\lambda/_D$**
  - Central spot called “Airy Disc”
- Every optical instrument
    - microscope, telescope, etc.
  - Each point on object
    - Produces Airy disc in image.

Diffraction-Limited Spot at the Focal Point of a Lens



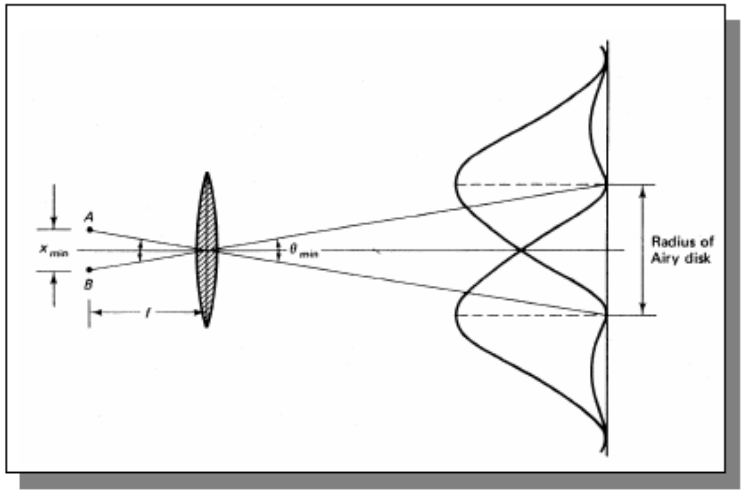
The width w of the Airy Disc at the focal point of a lens is given by: $w = \frac{1.22\lambda f}{D}$.

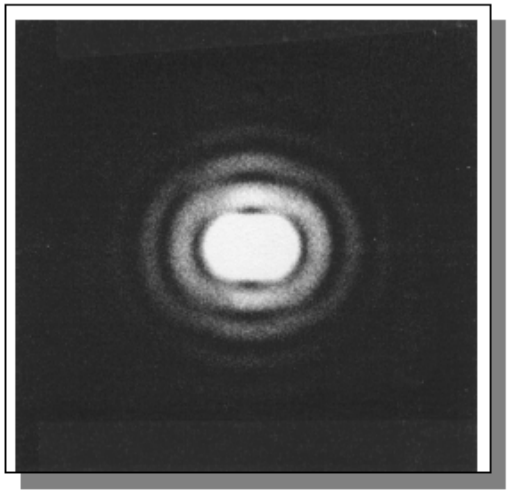
If ray aberrations in an optical system can be controlled such that all of the rays leaving a given object point land inside of the Airy Disc associated with the corresponding image point, we say we have Diffraction-Limited Imaging. (This is the absolute best you can do for an optical system that has lenses of a finite diameter!)

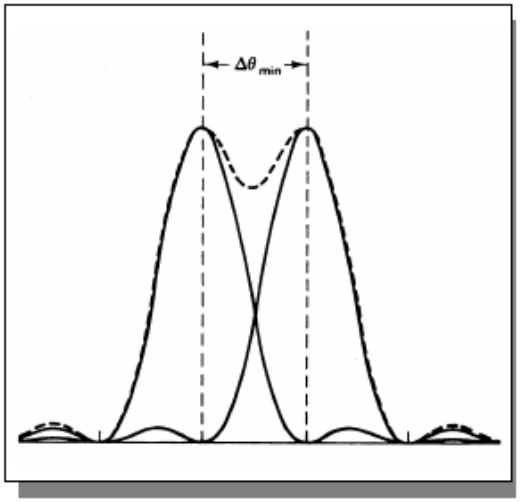
Diffraction limited resolution

- Diffraction sets the fundamental limit to the resolving power of optical instruments
- In practice, though, the image quality is often more seriously degraded by optical aberrations in the instrument

Rayleigh's criterion for resolution:







$$I(r_0) = I_0 \left[\frac{2J_1(kr_0 a / z)}{kr_0 a / z} \right]^2$$

$J_1(x) = 0$, when $x = 3.832$

$$\Delta\theta_{min} \approx \frac{r_0}{z} = 1.22 \frac{\lambda}{D} \quad ; \ (D=2a)$$

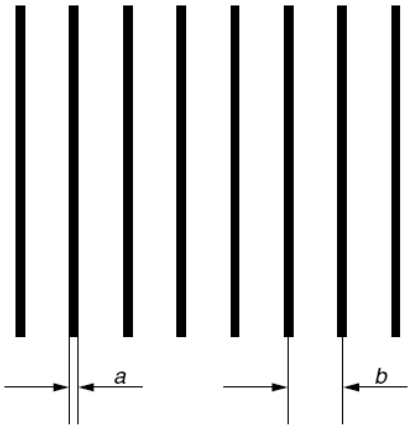
$$x_{min} = f\,\theta_{min}$$

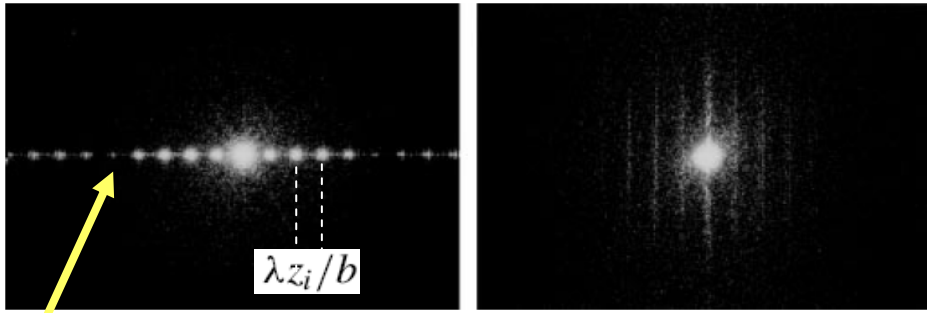
Infinite Array of Similar Apertures with Regular Spacing

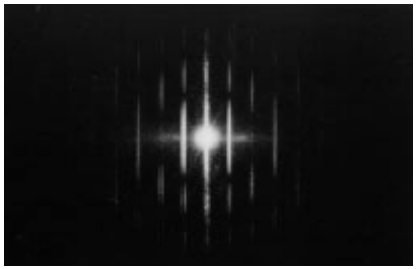
$$E(x_0, y_0) = \frac{1}{b}\Pi\left(\frac{x_0}{a}\right) * \Pi\left(\frac{x_0}{b}\right)$$

(Appendix : Step and repeat function)

$$\boxed{E(x_i, y_i) = a\ \mathrm{sinc}\left(\frac{a}{\lambda z_i}x_i\right) \cdot \Pi\left(\frac{b}{\lambda z_i}x_i\right)\delta\left(\frac{y_i}{\lambda z_i}\right)}$$







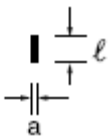
For $b/a = 5$ the fifth spike overlaps with the first null of $\mathrm{sinc}(a f_x)$ and the intensity of the fifth spike is faint.

Two-dimensional array of rectangular apertures

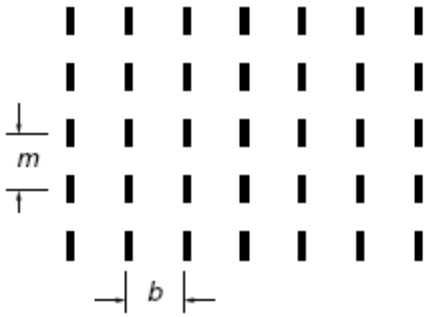
with dimensions $a \times l$, with period b in the x direction and m in the y direction

$$E(x_0, y_0) = \frac{1}{bm}\left[\Pi\left(\frac{x_0}{a}\right) * \text{Ш}\left(\frac{x_0}{b}\right)\right]\left[\Pi\left(\frac{y_0}{l}\right) * \text{Ш}\left(\frac{y_0}{m}\right)\right]$$

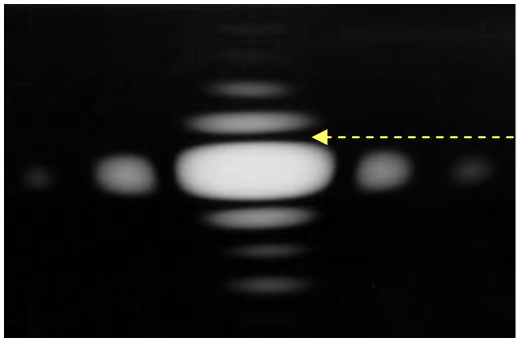
$$\boxed{E(x_i, y_i) = \frac{al}{j\lambda z_i}\,\text{sinc}\left(\frac{a}{\lambda z_i}x_i\right)\,\text{sinc}\left(\frac{l}{\lambda z_i}y_i\right)\text{Ш}\left(\frac{b}{\lambda z_i}x_i\right)\text{Ш}\left(\frac{m}{\lambda z_i}y_i\right)}$$

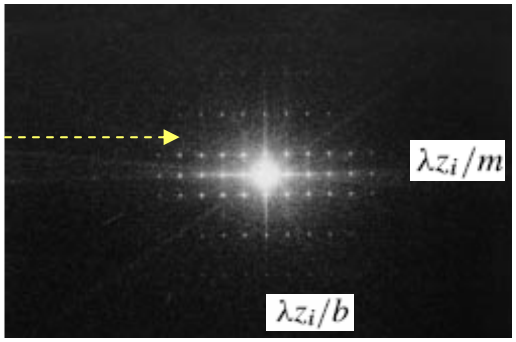


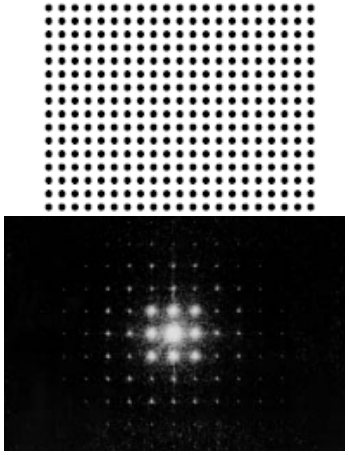
(1)



(2)







Two-dimensional array with irregular spacing

with dimensions $a \times l$, with period b in the x direction and m in the y direction

$$E(x_0, y_0) = \sum_{j=0}^{N-1} \Pi\left(\frac{x_0 - b_j}{a}\right) \Pi\left(\frac{y_0}{l}\right)$$ the x_0 direction by a random distance b_j from the origin

Using the shift theorem $\mathcal{F}\{g(x-a)\} = e^{-j2\pi f a} G(f)$

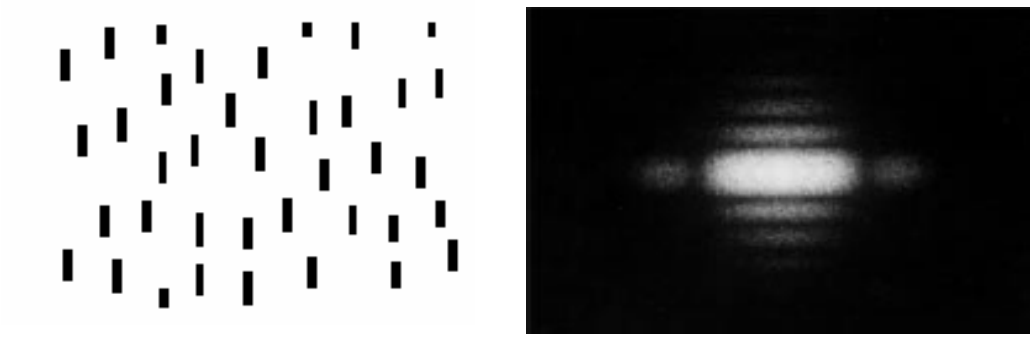
$$I(x_i, y_i) = E(x_i, y_i) \times E^*(x_i, y_i)$$

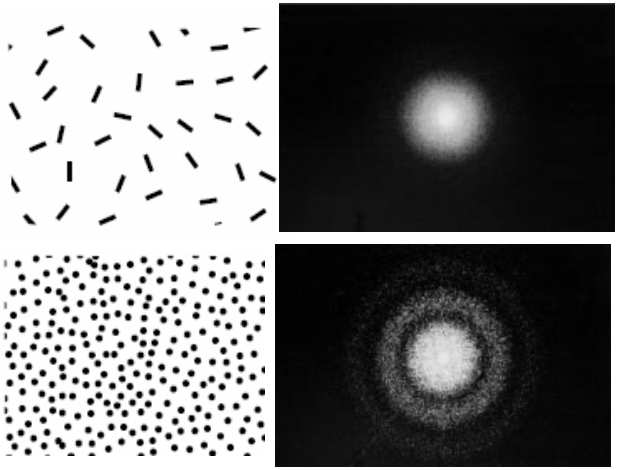
$$= E(x_i, y_i)(1 + e^{-j2\pi b_1 f_x} + e^{-j2\pi b_2 f_x} + \cdots) \times E^*(x_i, y_i)(1 + e^{j2\pi b_1 f_x} + e^{j2\pi b_2 f_x} + \cdots)$$

$$\boxed{= |E(x_i, y_i)|^2 \left(N + 2\sum_{j=1}^{N-1}\sum_{k=1}^{N-1} \cos 2 f_x (b_j - b_k)\right)}$$

diffraction pattern of a single rectangular aperture

randomized



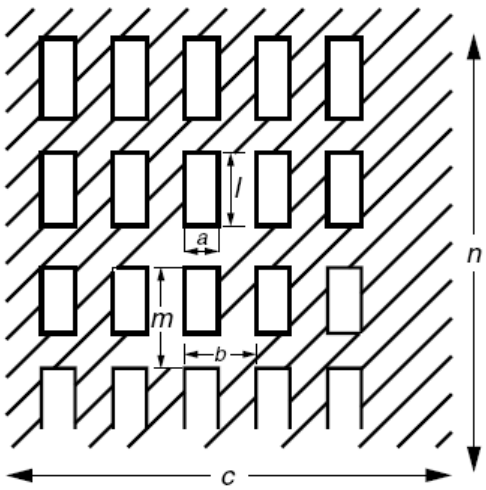


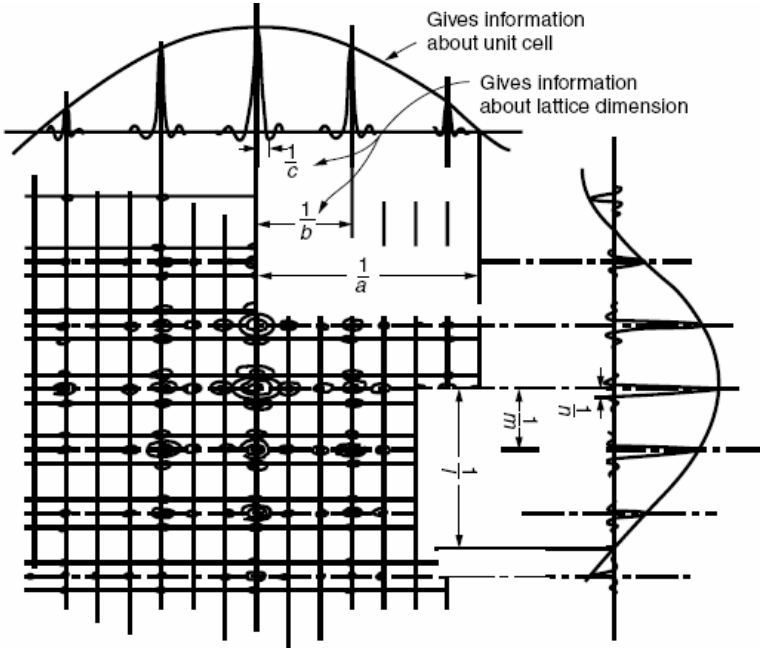
Diffraction from an array of finite size

$$E(x_0, y_0) = \frac{1}{bm}\left[\Pi\left(\frac{x_0}{a}\right) * \text{Ш}\left(\frac{x_0}{b}\right)\right]\Pi\left(\frac{x_0}{c}\right) \times \left[\Pi\left(\frac{y_0}{l}\right) * \text{Ш}\left(\frac{y_0}{m}\right)\right]\Pi\left(\frac{y_0}{n}\right)$$

$$\mathcal{F}\{E_0(x_0, y_0)\} = acln[\ \text{sinc}(af_x)\text{Ш}(bf_x)] * \ \text{sinc}(cf_x) \times [\ \text{sinc}(lf_y)\text{Ш}(mf_y)] * \ \text{sinc}(nf_y)$$

$a < b < c$





$$\frac{1}{c} < \frac{1}{b} < \frac{1}{a}$$

$1/c$ is the size of an individual spike
$1/b$ is the spacing between spots
$1/a$ is the overall size of the diffraction pattern

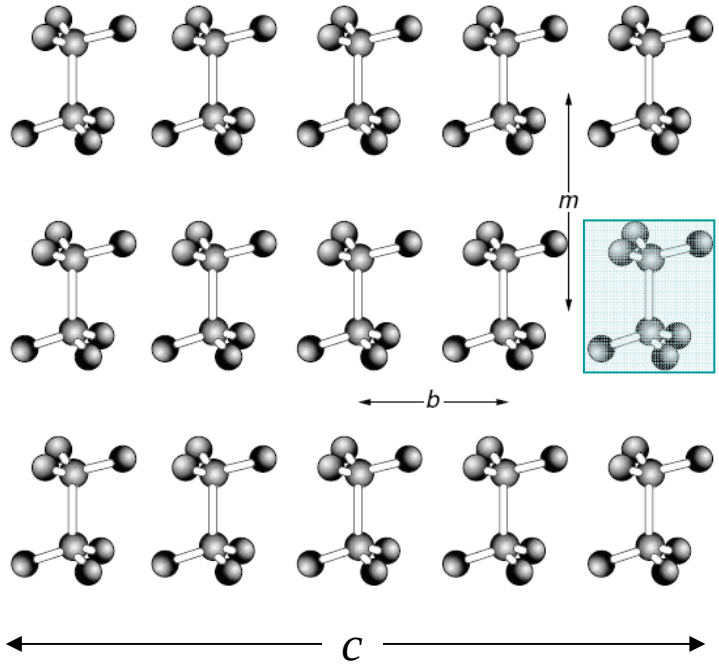
Diffraction from an array of finite size (2) : for large C

$$\mathcal{F}\{E_0(x_0, y_0)\} = acln[\ \mathrm{sinc}(af_x)\Pi(bf_x)] * \ \mathrm{sinc}(cf_x) \times [\ \mathrm{sinc}(lf_y)\Pi(mf_y)] * \ \mathrm{sinc}(nf_y)$$

For large c,

$$\mathcal{F}\{E_0(x_0, y_0)\} = acln\ \underline{\mathrm{sinc}(af_x)}\underline{[\Pi(bf_x) * \mathrm{sinc}(cf_x)]} \ \times \ \underline{\mathrm{sinc}(lf_y)}\underline{[\Pi(mf_y) * \ \mathrm{sinc}(nf_y)]}$$

(Pattern) = (Element pattern) × (array pattern)



unit cell (a x l)

The amplitude envelope function $G(f_x, f_y)$

$$G(f_x, f_y) = \mathcal{F}\{g(x_0, y_0)\}$$

contains information about the structure of the unit cell.

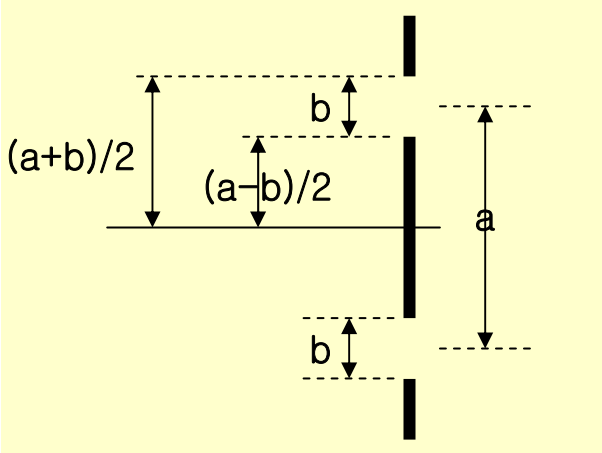
The structure of the unit cell $g(x_0, y_0)$ can be derived by

$$\mathcal{F}^{-1}|G(f_x, f_y)|^2 = g(x_0, y_0) * g(-x_0, -y_0)$$

Harker pattern

This is precisely the principle of X-ray crystallography.

Double-slit diffraction



$$E_R = \left(\frac{E_L}{r_o} \int_{-(a+b)/2}^{-(a-b)/2} e^{iks\sin\theta} ds \right) + \left(\frac{E_L}{r_o} \int_{(a-b)/2}^{(a+b)/2} e^{iks\sin\theta} ds \right)$$

$$= \frac{E_L}{r_o} \frac{1}{ik\sin\theta} \left[e^{\left(ik(-a+b)\sin\theta\right)/2} - e^{\left(ik(-a-b)\sin\theta\right)/2} + e^{\left(ik(a+b)\sin\theta\right)/2} - e^{\left(ik(a-b)\sin\theta\right)/2} \right]$$

$$\beta \equiv \tfrac{1}{2} kb\sin\theta, \quad \alpha \equiv \tfrac{1}{2} ka\sin\theta$$

$$E_R = \frac{E_L}{r_o} \frac{b}{2i\beta} \left[e^{i\alpha}(e^{i\beta} - e^{-i\beta}) + e^{-i\alpha}(e^{i\beta} - e^{-i\beta}) \right] = \frac{2E_L b}{r_o} \frac{\sin\beta}{\beta} \cos\alpha$$

$$I = \left(\frac{\varepsilon_o c}{2} \right) E_R^2 \qquad ;\ irradiance$$

$$= \left(\frac{\varepsilon_o c}{2} \right) \left(\frac{2E_L b}{r_o} \right)^2 \left(\frac{\sin\beta}{\beta} \right)^2 \cos^2\alpha = 4I_o \left(\frac{\sin\beta}{\beta} \right)^2 \cos^2\alpha$$

Double-slit diffraction

$$I = 4I_o\left(\frac{\sin\beta}{\beta}\right)^2\cos^2\alpha,\quad I_o = \left(\frac{\varepsilon_o c}{2}\right)\left(\frac{E_L b}{r_o}\right)^2$$

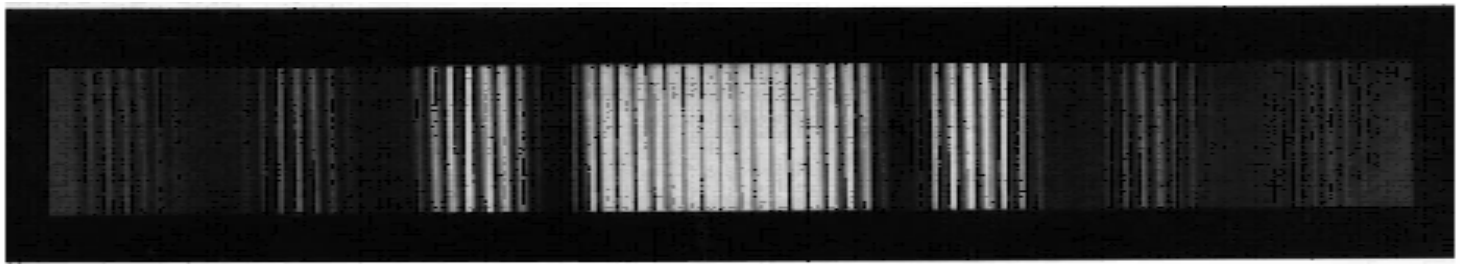
The interference term : $\cos^2\alpha = \cos^2\left[\frac{\pi a\sin\theta}{\lambda}\right]$ ---------- *(1)*

interference maxima : $p\lambda = a\sin\theta,\ \ p = 0, \pm 1, \pm 2, \cdots$

The diffraction envelope term : $\left(\frac{\sin\frac{\pi b\sin\theta}{\lambda}}{\frac{\pi b\sin\theta}{\lambda}}\right)^2$ ---------- *(2)*

diffraction minima : $m\lambda = b\sin\theta,\ \ m = 0, \pm 1, \pm 2, \cdots$

Condition for missing orders, or satisfying (1) & (2) : $a = \left(\frac{p}{m}\right)b$



Multiple-slit diffraction

$$E_R = \frac{E_L}{r_o} \sum_{j=1}^{N/2} \left\{ \int_{[-(2j-1)a-b)]/2}^{[-(2j-1)a+b)]/2} e^{iks\sin\theta} ds + \int_{[(2j-1)a-b)]/2}^{[(2j-1)a+b)]/2} e^{iks\sin\theta} ds \right\}$$

$$I = I_o \left(\frac{\sin\beta}{\beta} \right)^2 \left(\frac{\sin N\alpha}{\sin\alpha} \right)^2$$

The interference term : $\left(\frac{\sin N\alpha}{\sin\alpha} \right)^2$ ---------- (1)

principal maxima : $\alpha \to m\pi$, *or* $m\lambda = a\sin\theta$, $m = 0, \pm 1, \pm 2, \cdots$

magnitude : $\lim_{\alpha \to m\pi} \frac{\sin N\alpha}{\sin\alpha} = \lim_{\alpha \to m\pi} \frac{N\cos N\alpha}{\cos\alpha} = \pm N$

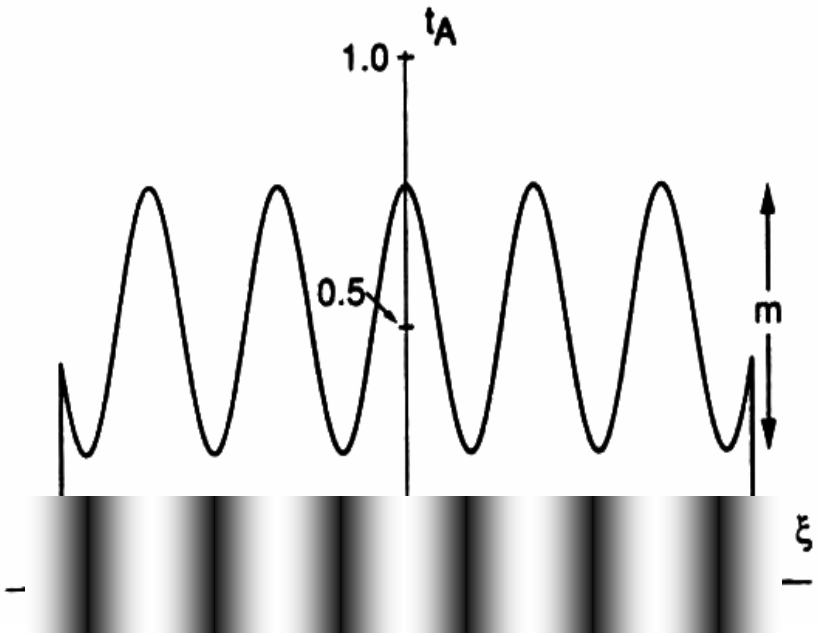
minima : $\alpha = \frac{p\pi}{N}$, *or* $\frac{p\lambda}{N} = a\sin\theta$, $p = 0, \pm 1, \pm 2, \cdots$

$\Rightarrow$ *principal maxima occur for* $p = 0, \pm N, \pm 2N, \cdots$

minima occur for $p =$ *all other integers*

Examples of Fraunhofer Diffraction

Thin sinusoidal amplitude grating



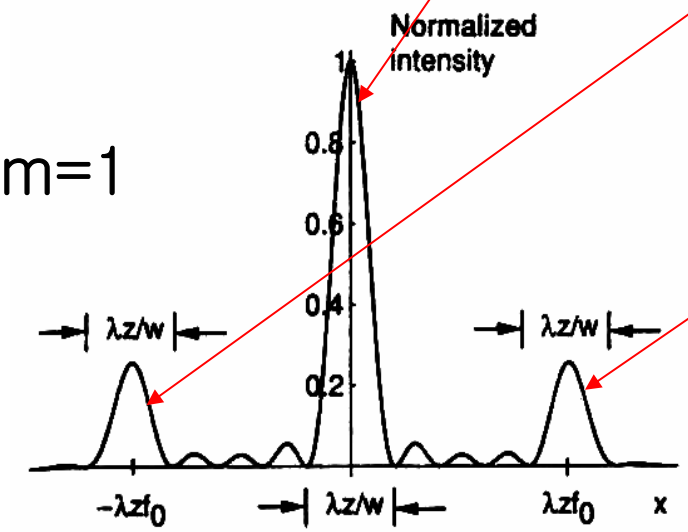
$$t_A = \left[\frac{1}{2} + \frac{m}{2}\cos(2\pi f_o \xi)\right] rect\left(\frac{\xi}{2\omega}\right) rect\left(\frac{\eta}{2\omega}\right)$$

Thin sinusoidal amplitude grating

$$FT\left[\frac{1}{2}+\frac{m}{2}\cos(2\pi f_o\xi)\right]=\frac{1}{2}\delta(f_X,f_Y)+\frac{m}{4}\delta(f_X+f_o,f_Y)+\frac{m}{4}\delta(f_X-f_o,f_Y)$$

$$I(x,y)=\left[\frac{A}{2\lambda z}\right]^2\sin c^2\left(\frac{2\omega y}{\lambda z}\right)\left\{\sin c^2\left(\frac{2\omega x}{\lambda z}\right)+\frac{m^2}{4}\sin c^2\left(\frac{2\omega}{\lambda z}(x+f_o\lambda z)\right)+\frac{m^2}{4}\sin c^2\left(\frac{2\omega}{\lambda z}(x-f_o\lambda z)\right)\right\}$$

m=1



$$\eta_0 = 0.25$$

$$\eta_{+1} = m^2/16 \le 6.25\%$$

$$\eta_{-1} = m^2/16 \le 6.25\%$$

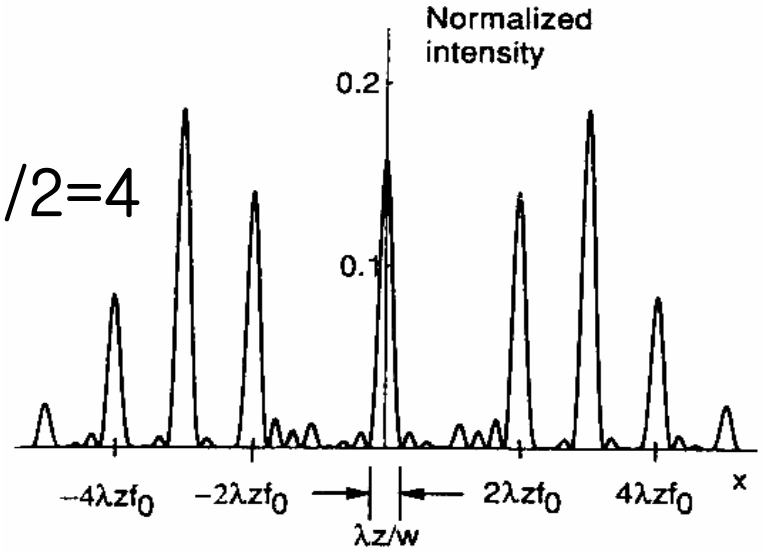
Thin sinusoidal phase grating

$$t_A(\xi,\eta) = \exp\left[j\frac{m}{2}\sin\left(2\pi f_0 \xi\right)\right] rect\left(\frac{\xi}{2w}\right) rect\left(\frac{\eta}{2w}\right)$$

$$FT\left\{\exp\left[j\frac{m}{2}\sin(2\pi f_o \xi)\right]\right\} = \sum_{q=-\infty}^{\infty} J_q\left(\frac{m}{2}\right)\delta(f_X - qf_o, f_Y)$$

$$I(x,y) \approx \left(\frac{A}{\lambda z}\right)^2 \sum_{q=-\infty}^{\infty} J_q^2\left(\frac{m}{2}\right) \mathrm{sinc}^2\left[\frac{2w}{\lambda z}(x - qf_0\lambda z)\right] \mathrm{sinc}^2\left(\frac{2wy}{\lambda z}\right)$$

m/2=4

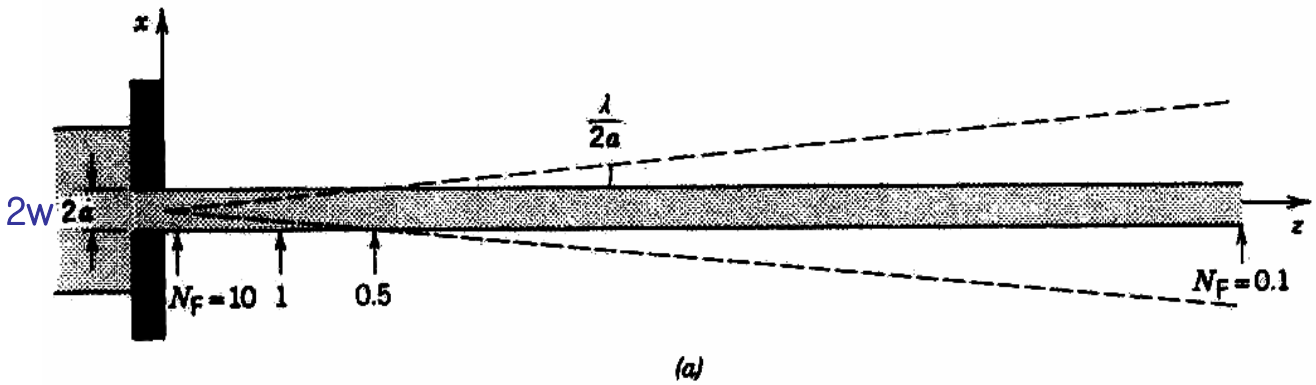


$$\eta_q = J_q^2\left(\frac{m}{2}\right)$$

$$\eta_{\pm 1,\max} = J_{\pm 1}^2\left(\frac{m}{2} = 1.8\right) = 33.8\%$$

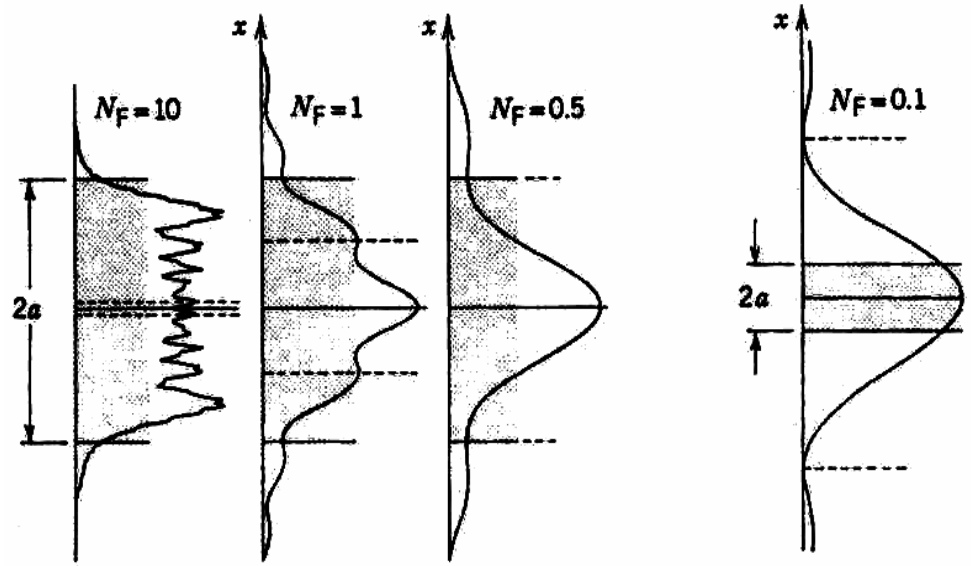
$$\eta_{0,\min} = J_0^2\left(\frac{m}{2} = 2.4\right) = 0, \text{ where } J_{\pm 1}^2\left(\frac{m}{2} = 2.4\right) = 27\%$$

Fresnel Diffraction by Square Aperture



(a)

Fresnel Diffraction from a slit of width $D = 2a$. (a) Shaded area is the geometrical shadow of the aperture. The dashed line is the width of the Fraunhofer diffracted beam.



(b)

(b) Diffraction pattern at four axial positions marked by the arrows in (a) and corresponding to the Fresnel numbers N_F=10, 1, 0.5, and 0.1. The shaded area represents the geometrical shadow of the slit. The dashed lines at $|x| = (\lambda / D)d$ represent the width of the Fraunhofer pattern in the far field. Where the dashed lines coincide with the edges of the geometrical shadow, the Fresnel number N_F=0.5.

$$N_F = w / \lambda z \ : \ Fresnel\ number$$

The Fresnel Formula

$$U_P = \frac{-ikU_0}{2\pi zz'} e^{ik|PS|} \int_A \exp\left(\frac{ik}{2z_a} \left[(x_0 - x_m)^2 + (y_0 - y_m)^2 \right] \right) dS$$

- let z' → ∞ and use a point on the axis x,y = 0

$$U_P = B \int_A \exp\left(\frac{ik}{2z} \left[x_0^2 + y_0^2 \right] \right) dS$$

$$= B \int_{x_1}^{x_2} \exp\left(\frac{ikx_0^2}{2z} \right) dx_0 \int_{y_1}^{y_2} \exp\left(\frac{iky_0^2}{2z} \right) dy_0$$

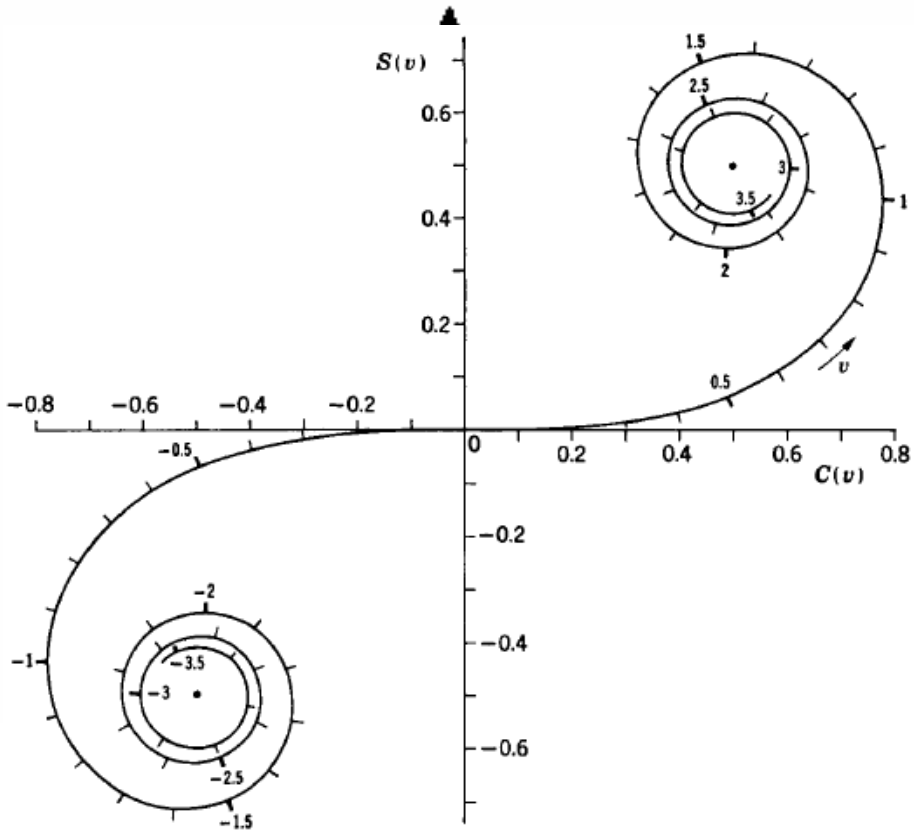
$$= B \int_{x_1}^{x_2} \exp\left(\frac{i\pi u^2}{2} \right) dx_0 \int_{y_1}^{y_2} \exp\left(\frac{i\pi v^2}{2} \right) dy_0$$

- where $u^2 = kx_0^2/(\pi z)$, $v^2 = ky_0^2/(\pi z)$

The Fresnel Integral

$$\int_{s_1}^{s_2} \exp(i\pi w^2/2)\,dw = \int_{s_1}^{s_2} \cos(\pi w^2/2)\,dw + i\int_{s_1}^{s_2} \sin(\pi w^2/2)\,dw$$
$$= C(s) + iS(s)$$

- Cornu spiral
- As s $\to \pm\infty$, C(s),S(s)$\to \pm$ ½
- Value of integral can be evaluated numerically



Fresnel Integral Definitions

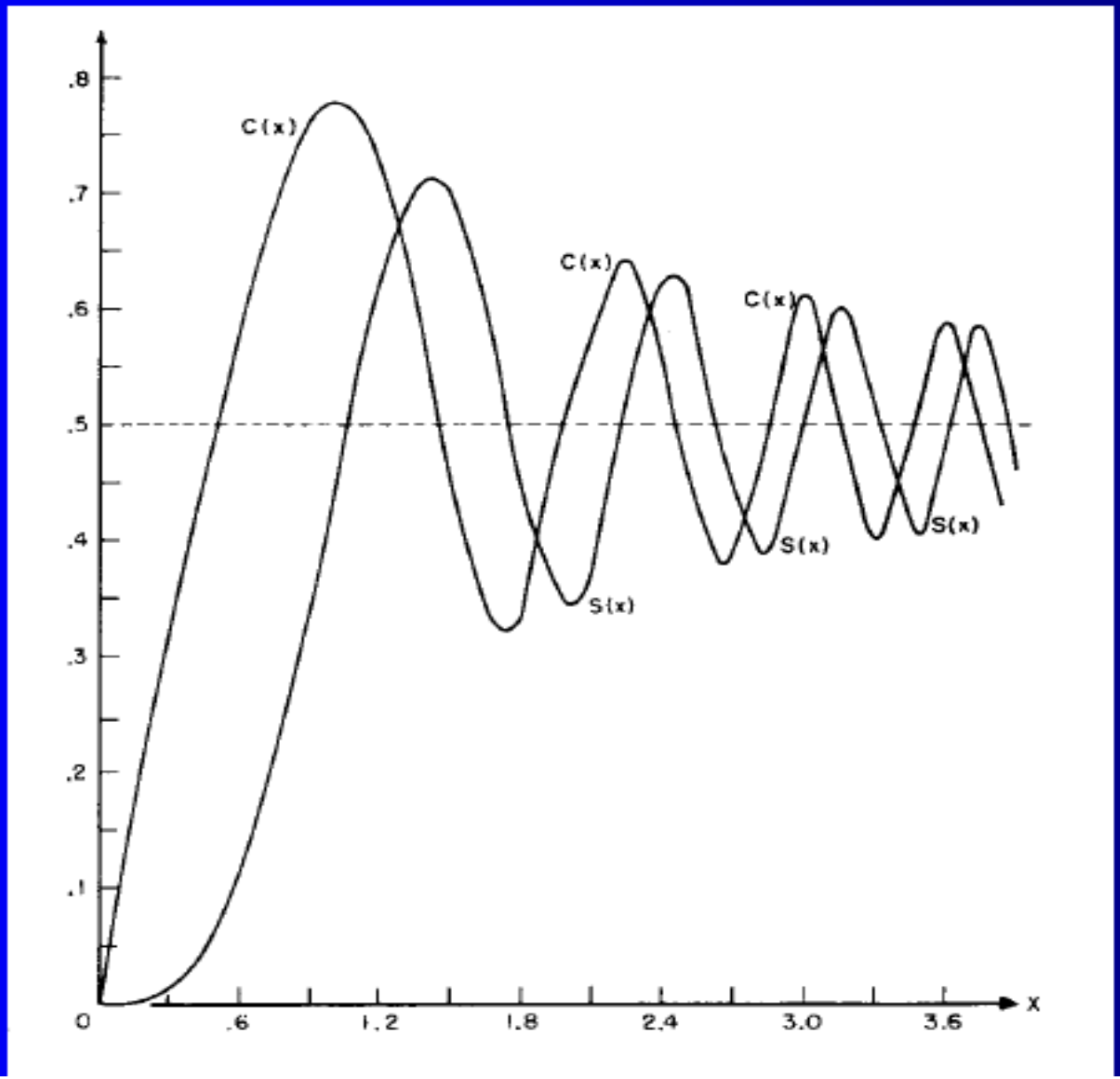
$$I(\xi) \equiv \int_0^{\xi} e^{+\frac{j\pi}{2}t^2}\, dt \quad \text{[Complex Fresnel Integral } I\text{]}$$

$$I(\xi) = C(\xi) + jS(\xi) \quad \text{[Fresnel Integrals } C \text{ and } S\text{]}$$

$$C(\xi) = \int_0^{\xi} \cos\left(\frac{\pi}{2}t^2\right) dt$$

$$S(\xi) = \int_0^{\xi} \sin\left(\frac{\pi}{2}t^2\right) dt$$

Plot of Fresnel Integrals



Fresnel Approximations

For $\xi \to 0$

$$e^{+\frac{j\pi}{2}t^2} \to 1 + j\frac{\pi}{2}\xi^2 - \frac{1}{2!}\left(\frac{\pi}{2}\right)^2 \xi^4 - j\frac{1}{3!}\left(\frac{\pi}{2}\right)^3 \xi^6 \mathrm{L}$$

$$\therefore I(\xi) = \int_0^{\xi} e^{+\frac{j\pi}{2}t^2} dt \to \xi + j\frac{1}{1!}\frac{\pi}{2}\frac{\xi^3}{3} - \frac{1}{2!}\left(\frac{\pi}{2}\right)^2 \frac{\xi^5}{5} - j\frac{1}{3!}\left(\frac{\pi}{2}\right)^3 \frac{\xi^7}{7}\mathrm{L}$$

$$C(\xi) = \int_0^{\xi} \cos\left(\frac{\pi}{2}t^2\right) dt \to \xi\left[1 - \frac{1}{2!5}\left(\frac{\pi}{2}\xi^2\right)^2 + \frac{1}{4!9}\left(\frac{\pi}{2}\xi^2\right)^4 \mathrm{L}\right]$$

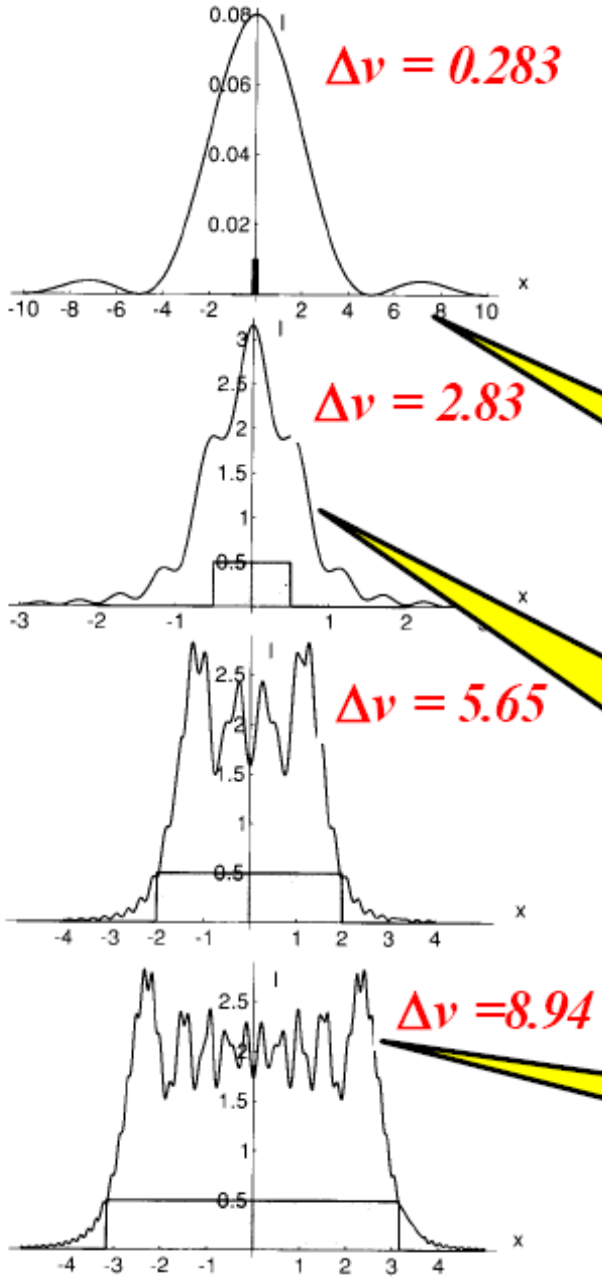
$$S(\xi) = \int_0^{\xi} \sin\left(\frac{\pi}{2}t^2\right) dt \to \xi\left[\frac{1}{1!3}\left(\frac{\pi}{2}\xi^2\right) - \frac{1}{3!7}\left(\frac{\pi}{2}\xi^2\right)^3 + \frac{1}{5!11}\left(\frac{\pi}{2}\xi^2\right)^5 \mathrm{L}\right]$$

For $\xi \to \infty$

$$C(\xi) \to \pm\frac{1}{2} + \frac{\sin(\pi\xi^2/2)}{\pi\xi}\mathrm{L} \qquad \text{for } z \gtrless 0$$

$$S(\xi) \to \pm\frac{1}{2} - \frac{\cos(\pi\xi^2/2)}{\pi\xi}\mathrm{L} \qquad \text{for } z \gtrless 0$$

Fresnel Slit Diffraction Evolution



Fourier transform-like result near far-zone

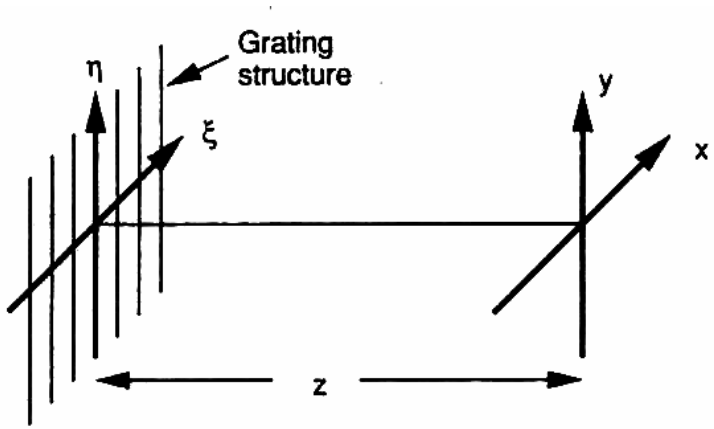
Sloughing off of diffraction "shoulders" with distance

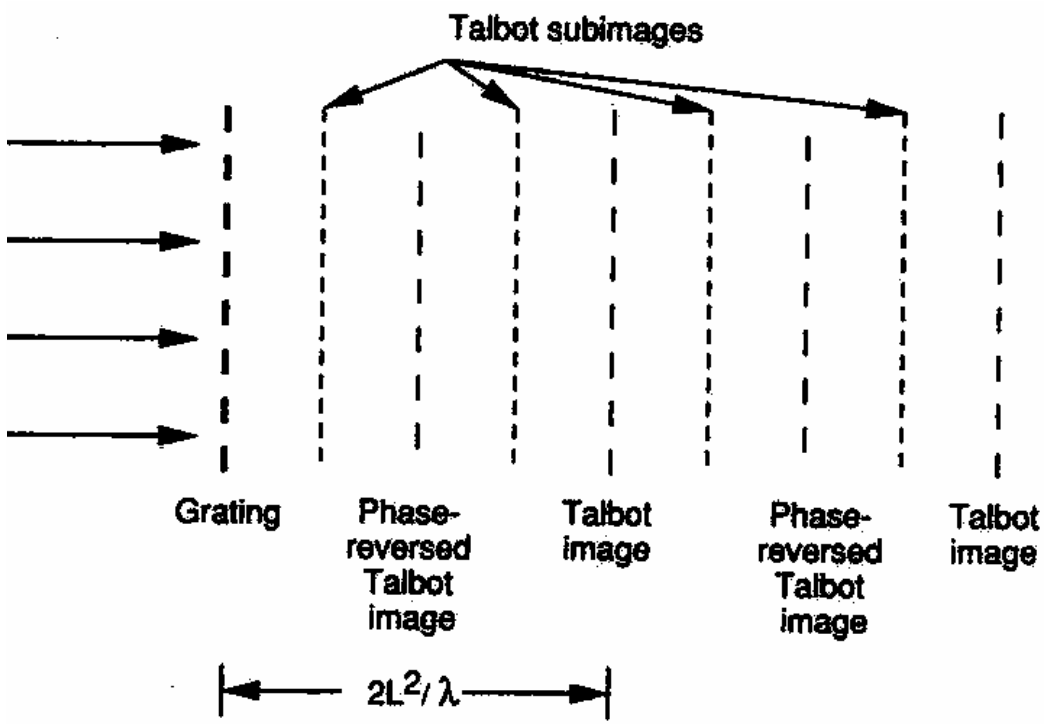
Geometric "image" of slit near aperture

Talbot Images

$$t_A(\xi,\eta)=\frac{1}{2}\left[1+m\cos(2\pi\xi/L)\right]$$

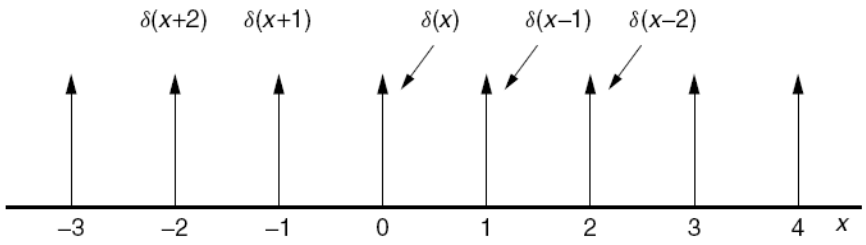
$$I(x,y)=\frac{1}{4}\left[1+2m\cos\left(\frac{\pi\lambda z}{L^2}\right)\cos\left(\frac{2\pi x}{L}\right)+m^2\cos^2\left(\frac{2\pi x}{L}\right)\right]$$





Appendix : Shah function (Impulse train function)

$$\text{Ш}(x) = \sum_{n=-\infty}^{\infty} \delta(x-n)$$



Fourier transform of the shah function

Because the shah function is a periodic function, it can be expanded into a Fourier series with a period of unity as

$$\text{Ш}(x) = \sum_{n=-\infty}^{\infty} a_n e^{j2\pi(n/T)x} \quad \text{where} \quad a_n = \frac{1}{T}\int_{-T/2}^{T/2} \delta(x) e^{-j2\pi(n/T)x}\, dx = 1$$

and where T is the period of the delta functions and is unity.

$$\text{Ш}(x) = \sum_{n=-\infty}^{\infty} e^{j2\pi n x}$$

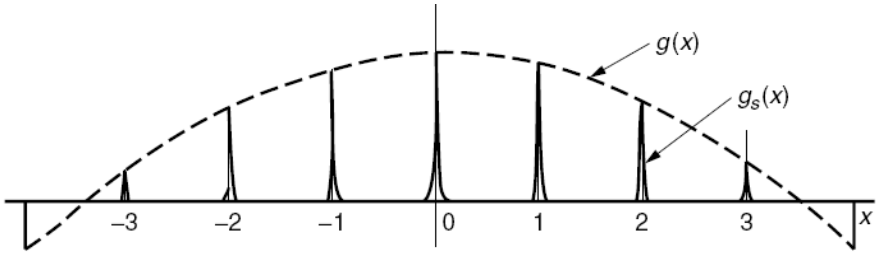
$$\mathcal{F}\{\text{Ш}(x)\} = \sum_{n=-\infty}^{\infty} \delta(f-n) \equiv \text{Ш}(f)$$

$$\mathcal{F}\{\text{Ш}(x/a)\} = a\text{Ш}(af)$$

Appendix : Sampling, Step and Repeat Function

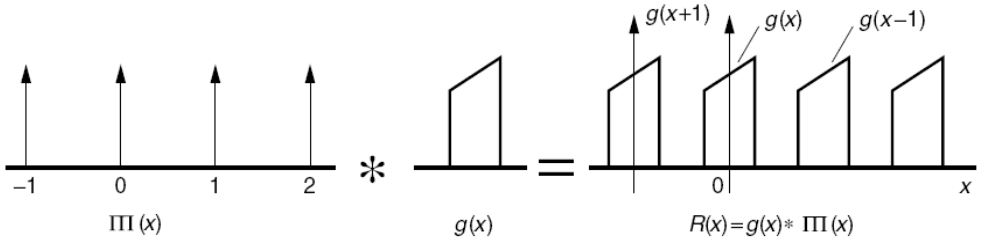
1. The generation of a sampled function

$$g_s(x) = g(x)\Pi(x)$$



2. The generation of a step and repeat function by convolving $g(x)$ with the shah function:

$$R(x) = g(x) * \Pi(x)$$
$$= \int_{-\infty}^{\infty} g(\tau)\Pi(x-\tau)\,d\tau$$
$$= \int_{-\infty}^{\infty} g(\tau) \sum_{n=-\infty}^{\infty} \delta(x-\tau-n)\,d\tau = \sum_{n=-\infty}^{\infty} g(x-n)$$



The step and repeat function at an interval of a

$$R(x) = \frac{1}{a} g(x) * \Pi\left(\frac{x}{a}\right)$$