

Optical properties of materials

Insulating media (dielectric) : Lorentz model

Conducting media (Metal, in free-electron region) : Drude model

Conducting media (Metal, in bound-electron region) : Drude-Sommerfeld model

→ Extended Drude model (Lorentz-Drude model)

$$D(\omega) = \epsilon_0 \epsilon_r(\omega) E(\omega) \quad (\text{spatially local response of media})$$

$\epsilon_r(\omega)$: relative dielectric constant
= relative permittivity
= dielectric function

Let's start from Maxwell equations

Maxwell's Equations

Divergence Equations $\nabla \cdot \mathbf{D} = \rho$

$$\nabla \cdot \mathbf{B} = 0$$

Curl Equations $\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

\mathbf{D} = Electric flux density

\mathbf{B} = Magnetic flux density

\mathbf{E} = Electric field vector

\mathbf{H} = Magnetic field vector

ρ = charge density (external)

\mathbf{J} = current density (external)

Flux densities in a medium

→ Total electric flux density = Flux from external field + flux due to material polarization

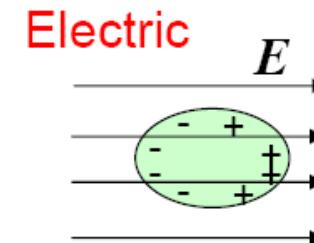
Electric $\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}(E) = \epsilon \mathbf{E}$

When \mathbf{P} is proportional to \mathbf{E}

Electric polarization vector..... Material dependent!!

$$\epsilon_0 = 8.85 \cdot 10^{-12} \text{ C}^2 \text{N}^{-1} \text{m}^{-2} [\text{F/m}]$$

ϵ = permittivity of material



Magnetic $\mathbf{B} = \mu_0 \mathbf{H} + \mu_0 \mathbf{M}(H)$

$\mathbf{M} = 0 \Rightarrow \mathbf{B} = \mu_0 \mathbf{H}$ in most of optical materials.

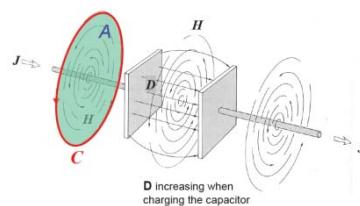
Magnetic field vector Magnetic polarization vector

$$\mu_0 = \text{permeability of free space} = 4\pi \times 10^{-7} \text{ H/m}$$

Divergence Equations

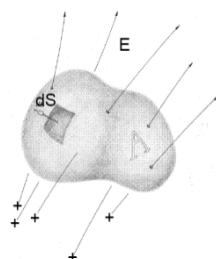
Gauss's Law (Gauss 1777-1855)

$$\int_A \mathbf{D} \cdot d\mathbf{S} = \int_A \epsilon E \cdot d\mathbf{S} = \int_V \rho dv$$



Gauss's Theorem (very general)

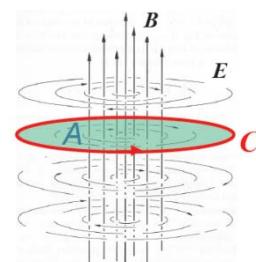
$$\int_A \mathbf{F} \cdot d\mathbf{S} = \int_V \nabla \cdot \mathbf{F} dv$$



→ $\nabla \cdot \mathbf{D} = \rho$

→ $\nabla \cdot \mathbf{B} = 0$ is derived in a similar way from $\int_A \mathbf{B} \cdot d\mathbf{S} = 0$

$$\nabla \times \nabla \times \mathbf{E} = \nabla (\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}$$



Curl Equations

Ampere-Maxwell's Law

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_A \left(\frac{\partial \mathbf{D}}{\partial t} + \mathbf{J} \right) \cdot d\mathbf{S}$$

Stokes theorem (very general)

$$\oint_C \mathbf{F} \cdot d\mathbf{l} = \int_A (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$$

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_A (\nabla \times \mathbf{H}) \cdot d\mathbf{S} = \int_A \left(\frac{\partial \mathbf{D}}{\partial t} + \mathbf{J} \right) \cdot d\mathbf{S}$$

→ $\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}$

Faraday's Law (Faraday 1775-1836)

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = - \int_A \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}$$

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = \int_A (\nabla \times \mathbf{E}) \cdot d\mathbf{S} = - \int_A \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}$$

→ $\nabla \times \mathbf{E} = - \frac{\partial \mathbf{B}}{\partial t}$

EM wave Equation in bulk media (ϵ, μ)

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = -\mu_0 \frac{\partial \mathbf{H}}{\partial t} \text{ (Materials with } \mathbf{M} = 0 \text{ only)}$$

$$\nabla \times \nabla \times \mathbf{E} = \nabla \times \left(-\mu_0 \frac{\partial \mathbf{H}}{\partial t} \right) = -\mu_0 \frac{\partial (\nabla \times \mathbf{H})}{\partial t}$$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J} \quad \rightarrow \quad \nabla \times \nabla \times \mathbf{E} = -\mu_0 \frac{\partial^2 \mathbf{D}}{\partial t^2} - \mu_0 \frac{\partial \mathbf{J}}{\partial t}$$

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \quad \rightarrow \quad \nabla \times \nabla \times \mathbf{E} = -\mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} - \mu_0 \frac{\partial^2 \mathbf{P}}{\partial t^2} - \mu_0 \frac{\partial \mathbf{J}}{\partial t}$$

$$\nabla \times \nabla \times \mathbf{E} = \nabla (\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}$$

$$\nabla \cdot \mathbf{D} = \nabla \cdot \epsilon \mathbf{E} = \epsilon \nabla \cdot \mathbf{E} + \mathbf{E} \cdot \nabla \epsilon \quad \nabla \cdot \mathbf{E} = 0 \text{ when 1) } \rho_f = 0$$

2) $\epsilon(r)$ does not vary significantly within a λ distance

➡
$$\nabla^2 \mathbf{E} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} + \mu_0 \frac{\partial^2 \mathbf{P}}{\partial t^2} + \mu_0 \frac{\partial \mathbf{J}}{\partial t}$$
 : EM wave equation in bulk media without net charge
($\mu = \mu_0$ and $\rho = 0$)

EM waves in dispersive media : $\epsilon = \epsilon(\omega)$, $\mu = \mu(\omega)$

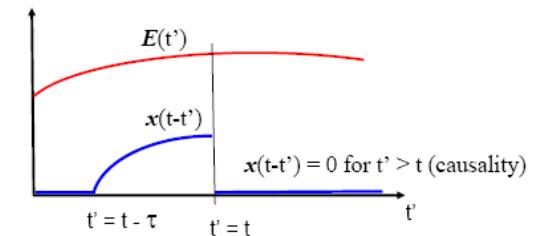
$$\nabla^2 \mathbf{E} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} + \mu_0 \frac{\partial^2 \mathbf{P}}{\partial t^2} + \mu_0 \frac{\partial \mathbf{J}}{\partial t}$$

Relation between \mathbf{E} and \mathbf{P} (\mathbf{J}) is dynamic:

In ideal case, assuming an instantaneous response, $\longrightarrow P(r, t) = \epsilon_0 \chi E(r, t)$

But, in real life, $P(r, t) = \epsilon_0 \int_{-\infty}^{+\infty} dt' x(t-t') E(r, t')$ \longrightarrow **Response to E is DISPERSIVE**

$P(r, t)$ results from response to E over some characteristic time τ :
Function $x(t)$ is a scalar function lasting a characteristic time τ



Convolution theorem

$$P(k, \omega) = \underline{\epsilon_0 \chi(\omega)} E(k, \omega) \quad \xrightarrow{\text{Convolution theorem}} \quad \epsilon(\omega) = \epsilon_0 [1 + \chi(\omega)]$$

→ Therefore, the temporally dispersive response of a medium induces a frequency dependence in permittivity.

A. Propagation in insulating media (dielectric : $P, J = 0$)

Consider a linear, homogeneous, isotropic medium .

$$\mathbf{P} \text{ linearly proportional to } \mathbf{E}: \quad \mathbf{P} = \epsilon_0 \chi \mathbf{E}$$

$$\mathbf{D} = \epsilon \mathbf{E} = \epsilon_0 \mathbf{E} + \mathbf{P} = \epsilon_0 + \epsilon_0 \chi \mathbf{E} = \epsilon_0 (1 + \chi) \mathbf{E} = \epsilon_r \epsilon_0 \mathbf{E}$$

χ is a scalar constant called the “electric susceptibility”

Note 1 : In anisotropic media \mathbf{P} and \mathbf{E} are not necessarily parallel: $P_i = \sum_j \epsilon_0 \chi_{ij} E_j$

Note2 : In non-linear media: $\mathbf{P} = \epsilon_0 \chi \mathbf{E} + \epsilon_0 \chi^{(2)} \mathbf{E}^2 + \epsilon_0 \chi^{(3)} \mathbf{E}^3 + \dots$

Note 3 : In inhomogeneous media: $\nabla \cdot \mathbf{D} = \nabla \cdot \epsilon \mathbf{E} = \epsilon \nabla \cdot \mathbf{E} + \mathbf{E} \cdot \nabla \epsilon$

$$\nabla^2 \mathbf{E} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} + \mu_0 \frac{\partial^2 \mathbf{P}}{\partial t^2} + \cancel{\mu_0 \frac{\partial \mathbf{J}}{\partial t}} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} + \mu_0 \epsilon_0 \chi \frac{\partial^2 \mathbf{E}}{\partial t^2} = \mu_0 \epsilon_0 (1 + \chi) \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

Define relative dielectric constant as: $\epsilon_r = 1 + \chi$: all the materials properties resulting from \mathbf{P}

→
$$\nabla^2 \vec{E} = \mu_0 \epsilon_0 \epsilon_r \frac{\partial^2 \vec{E}}{\partial t^2} = \epsilon_r \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2}$$
 : EM wave equation in insulating media (dielectric)
(linear, homogeneous, isotropic, $J=0$, $\mu=\mu_0$, and $\rho=0$)

The wave equation is satisfied by an electric field of the form of:

$$\vec{E} = \vec{E}_0 \exp[i(\vec{k} \cdot \vec{r} - \omega t)] \longrightarrow k^2 = \epsilon_r \frac{\omega^2}{c^2} \text{ (dispersion relation)}, \text{ where } \epsilon_r = 1 + \chi(\omega)$$

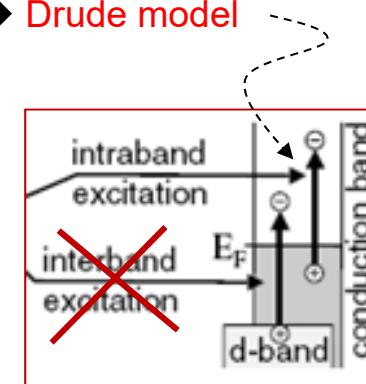
B. Propagation in conducting media (free-electron region: $J, P = 0$)

Consider an *ideally free electron* (not bounded to a particular nucleus) → Drude model

$$J \text{ linearly proportional to } E: \quad J = \sigma E$$

σ is the conductivity

$$\nabla^2 E = \mu_0 \epsilon_0 \frac{\partial^2 E}{\partial t^2} + \mu_0 \cancel{\frac{\partial^2 P}{\partial t^3}} + \mu_0 \frac{\partial J}{\partial t} = \mu_0 \epsilon_0 \frac{\partial^2 E}{\partial t^2} + \mu_0 \frac{\partial J}{\partial t}$$



→
$$\nabla^2 \vec{E} = \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} + \frac{\sigma}{\epsilon_0 c^2} \frac{\partial \vec{E}}{\partial t}$$
 : EM wave equation in conducting media (free-electron region)
(linear, homogeneous, isotropic, $P=0$, $\mu=\mu_0$, and $\rho=0$)

The wave equation is satisfied by an electric field of the form of:

$$\vec{E} = \vec{E}_0 \exp \left[i(\vec{k} \cdot \vec{r} - \omega t) \right] \longrightarrow k^2 = \left[1 + i \frac{\sigma}{\epsilon_0 \omega} \right] \frac{\omega^2}{c^2} = \epsilon_r(\omega) \frac{\omega^2}{c^2} \text{ (dispersion relation)}$$

$$\epsilon_r(\omega) = 1 + i \frac{\sigma(\omega)}{\epsilon_0 \omega} \quad [\epsilon_r = 1 + \chi(\omega) \text{ for dielectric}]$$

C. Propagation in conducting media (bound-electron region: $J, P \neq 0$)

Consider a bounded electron (not ideal free-electron due to *interband transition*)

→ Modified Drude model

→ Quasi-free-electron model

→ Drude-Sommerfeld model

For the noble metals* (e.g. Au, Ag, Cu), the filled d-band close to Fermi surface causes *a residual polarization*, P_b , due to the positive background of the ion cores.

*In physics the definition of a noble metal is strict. It is required that the d-bands of the electronic structure are filled.

Taking this into account, **only copper, silver and gold are noble metals**, as all d-like bands are filled and don't cross the Fermi level.

$Cu^{29} : 1s^2 \dots 3d^{10} \ 4s^1$

$Ag^{47} : 1s^2 \dots 4d^{10} \ 5s^1$

$Au^{79} : 1s^2 \dots 5d^{10} \ 6s^1$

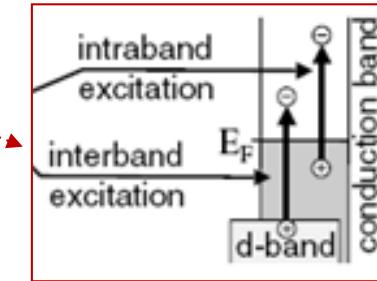
$Al^{13} : 1s^2 \dots 3s^2 \ 3p^1$ (different)

**Interband transition* : excitation of electrons from deeper bands into the conduction band

$$P_b \text{ also linearly proportional to } E: \quad P_b = \epsilon_0 \chi_b E$$

$$P_b \text{ induces internal current density: } J_b = \frac{\partial P_b}{\partial t} = \epsilon_0 \chi_b \frac{\partial E}{\partial t}$$

$$\nabla^2 \vec{E} = \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} + \mu_0 \frac{\partial^2 \vec{P}_b}{\partial t^2} + \mu_0 \frac{\partial \vec{J}}{\partial t} = \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} + \mu_0 \frac{\partial}{\partial t} (\vec{J}_b + \vec{J})$$



→ $\nabla^2 \vec{E} = \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} + \frac{\chi_b}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} + \frac{\sigma}{\epsilon_0 c^2} \frac{\partial \vec{E}}{\partial t}$: EM wave equation in conducting media (bound-electron region)
(linear, homogeneous, isotropic, $\mu=\mu_0$, and $\rho=0$)

The wave equation is satisfied by an electric field of the form of:

$$\vec{E} = \vec{E}_0 \exp \left[i(\vec{k} \cdot \vec{r} - \omega t) \right] \longrightarrow k^2 = \left[1 + \chi_b + i \frac{\sigma}{\epsilon_0 \omega} \right] \frac{\omega^2}{c^2} = \epsilon_r(\omega) \frac{\omega^2}{c^2}$$

$$\epsilon_r(\omega) = \epsilon_\infty + i \frac{\sigma(\omega)}{\epsilon_0 \omega} \quad [\epsilon_\infty = 1 + \chi_b(\omega)]$$

Now, how to determine $\chi(\omega)$, $\sigma(\omega)$, and $\chi_b(\omega)$?

Microscopic origin of ω -response of matter

Professor Vladimir M. Shalaev, Univ of Purdue

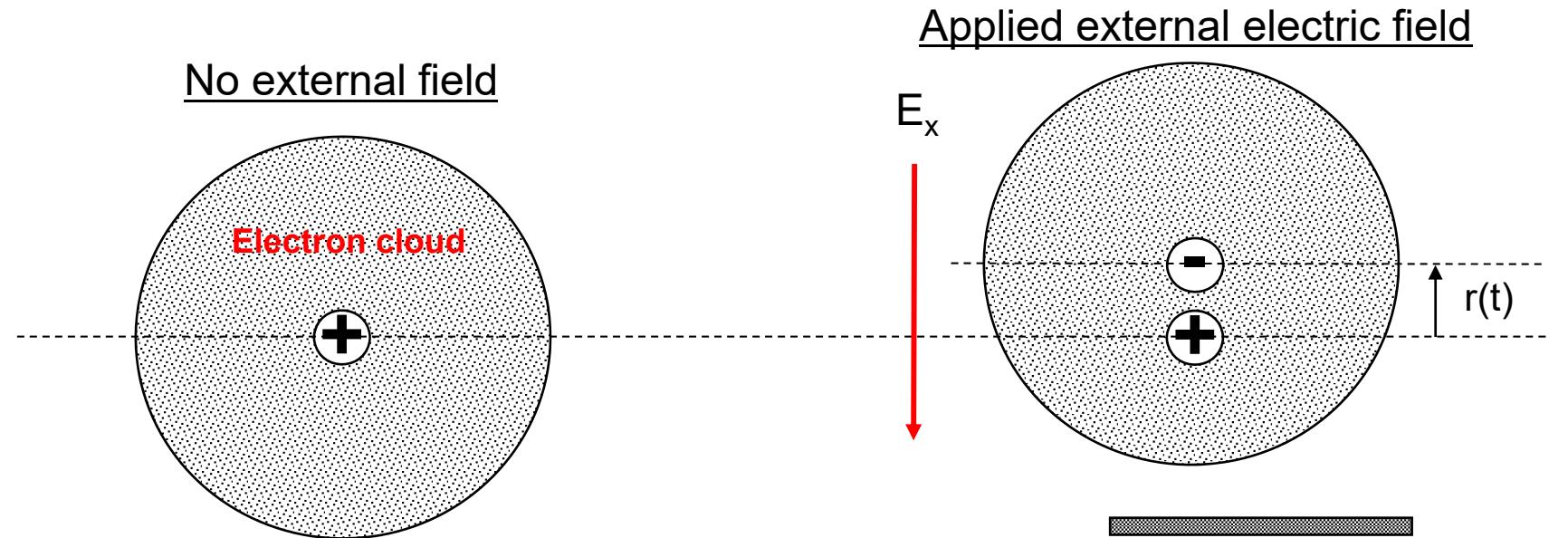
Origin frequency dependence of χ in real materials

- Lorentz model (harmonic oscillator model)
- Insulators (Lattice absorption, color centers...)
- Semiconductors (Energy bands, Urbach tail, excitons ...)
- Metals (AC conductivity, Plasma oscillations, interband transitions...)

Real and imaginary part of χ are linked

- Kramers-Kronig

Classical Electron Oscillator (CEO) Model = Lorentz model



The equation of motion of the oscillating electron,

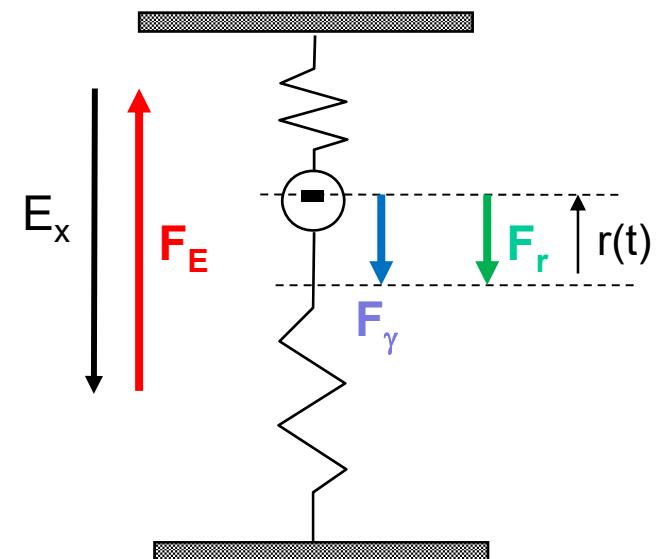
$$m \frac{d^2 \vec{r}}{dt^2} = \vec{F}_r(\vec{r}) + \vec{F}_\gamma(\vec{v}) + \vec{F}_E(\vec{E}) = -C\vec{r} - m\gamma \frac{d\vec{r}}{dt} - e\vec{E}_x$$

m : effective mass of each electron

$\gamma = \frac{1}{\tau}$: damping constant (or, collision frequency)

(τ is known as the relaxation time of the free electron gas)

(γ is typically 100 THz = 10^{14} Hz)



A. $\chi(\omega)$ of insulating media (dielectric : $P, J = 0$)

Behavior of bound electrons in an electromagnetic field

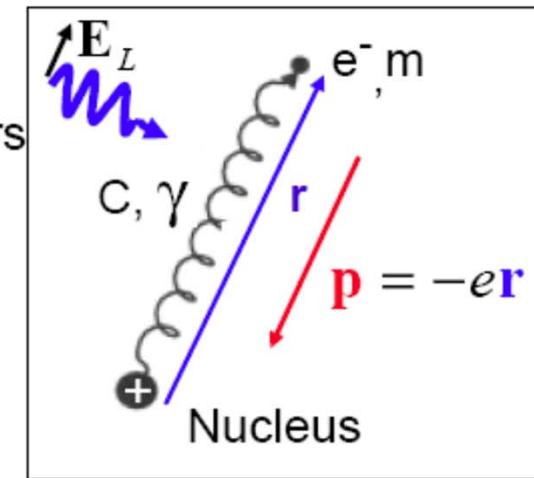
- Optical properties of insulators are determined by bound electrons

Lorentz model

- Charges in a material are treated as harmonic oscillators

$$m\mathbf{a}_{el} = \mathbf{F}_{E,Local} + \mathbf{F}_{Damping} + \mathbf{F}_{Spring} \quad (\text{one oscillator})$$

$$m \frac{d^2\mathbf{r}}{dt^2} + m\gamma \frac{d\mathbf{r}}{dt} + C\mathbf{r} = -e\mathbf{E}_L \exp(-i\omega t)$$



- The electric dipole moment of this system is: $\mathbf{p} = -e\mathbf{r}$

$$m \frac{d^2\mathbf{p}}{dt^2} + m\gamma \frac{d\mathbf{p}}{dt} + C\mathbf{p} = e^2\mathbf{E}_L \exp(-i\omega t)$$

- Guess a solution of the form:

$$\mathbf{p} = \mathbf{p}_0 \exp(-i\omega t) ; \frac{d\mathbf{p}}{dt} = -i\omega \mathbf{p}_0 \exp(-i\omega t) ; \frac{d^2\mathbf{p}}{dt^2} = -\omega^2 \mathbf{p}_0 \exp(-i\omega t)$$

$$\Rightarrow -m\omega^2 \mathbf{p}_0 - i\omega m\gamma \mathbf{p}_0 + C\mathbf{p}_0 = e^2\mathbf{E}_L \Rightarrow \text{Solve for } \mathbf{p}_0(\mathbf{E}_L)$$

Atomic Polarizability

Determination of atomic polarizability

- Last slide: $-m\omega^2 \mathbf{p}_0 - im\gamma\omega \mathbf{p}_0 + C \mathbf{p}_0 = e^2 \mathbf{E}_L$

$\Rightarrow -\omega^2 \mathbf{p}_0 - i\gamma\omega \mathbf{p}_0 + \frac{C}{m} \mathbf{p}_0 = \frac{e^2}{m} \mathbf{E}_L$ (Divide by m)

Define as ω_0^2 (turns out to be the resonance ω)

$\Rightarrow \mathbf{p}_0 = \frac{e^2}{m} \frac{1}{\omega_0^2 - \omega^2 - i\gamma\omega} \mathbf{E}_L$

Atomic polarizability (in SI units)

- Define atomic polarizability: $\alpha(\omega) \equiv \frac{p_0}{\epsilon_0 E_L} = \frac{e^2}{\epsilon_0 m} \frac{1}{\omega_0^2 - \omega^2 - i\gamma\omega}$

↑
↑

Resonance frequency
Damping term

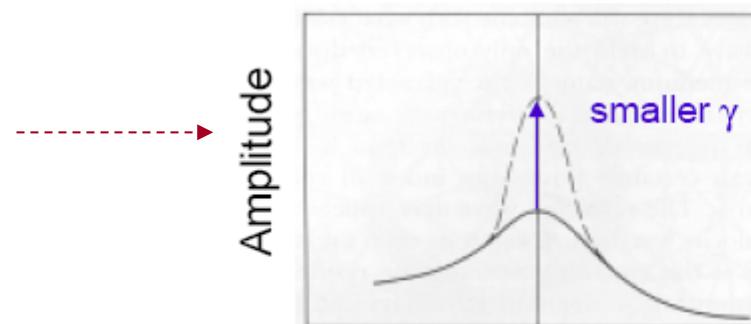
Characteristics of the Atomic Polarizability

Response of matter (\mathbf{P}) is not instantaneous $\Rightarrow \omega$ -dependent response

- Atomic polarizability: $\alpha(\omega) = \frac{P_0}{\epsilon_0 E_L} = \frac{e^2}{\epsilon_0 m} \frac{1}{\omega_0^2 - \omega^2 - i\gamma\omega} = A \exp[i\theta(\omega)]$

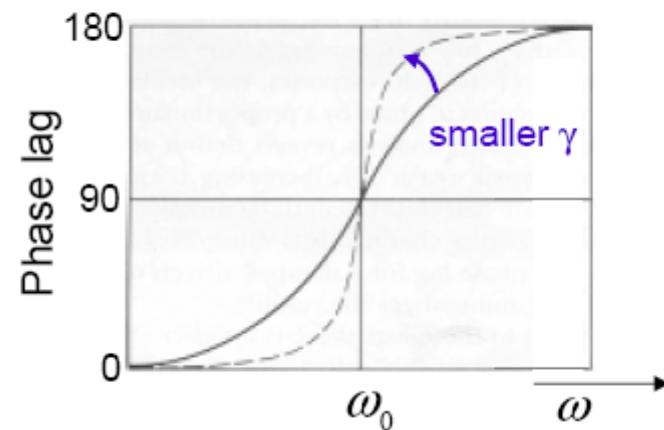
- Amplitude

$$A = \frac{e^2}{\epsilon_0 m} \frac{1}{\left[(\omega_0^2 - \omega^2)^2 + \gamma^2 \omega^2 \right]^{1/2}}$$



- Phase lag of α with E :

$$\theta = \tan^{-1} \frac{\gamma\omega}{\omega_0^2 - \omega^2}$$



Relation Atomic Polarizability (α) and χ : 2 cases

Case 1: Rarified media (.. gasses)

- Dipole moment of one atom, j :

$$\mathbf{p}_j = \epsilon_0 \alpha_j(\omega) \mathbf{E}_L$$

E-field photon

- Polarization vector:

↑
Occurs in Maxwell's equation..

$$\mathbf{P} = \frac{1}{V} \sum_j \mathbf{p}_j = \frac{\epsilon_0}{V} \sum_j \alpha_j \mathbf{E}_L = \epsilon_0 N \alpha_j \mathbf{E}_L$$

Density
sum over all atoms

$$\alpha_j(\omega) = \frac{e^2}{\epsilon_0 m} \frac{1}{\omega_0^2 - \omega^2 - i\gamma\omega}$$

$$\Rightarrow \mathbf{P} = \frac{Ne^2}{m} \frac{1}{\omega_0^2 - \omega^2 - i\gamma\omega} \mathbf{E}_L \quad (= \epsilon_0 \chi \mathbf{E}_L) \quad \Rightarrow \therefore \chi(\omega) = N \alpha_j$$

$$\Rightarrow \text{Microscopic origin susceptibility: } \chi(\omega) = \frac{Ne^2}{\epsilon_0 m} \frac{1}{\omega_0^2 - \omega^2 - i\gamma\omega}$$

- Plasma frequency defined as: $\omega_p^2 = \frac{Ne^2}{\epsilon_0 m}$

$$\Rightarrow \chi(\omega) = \frac{\omega_p^2}{\omega_0^2 - \omega^2 - i\gamma\omega}$$

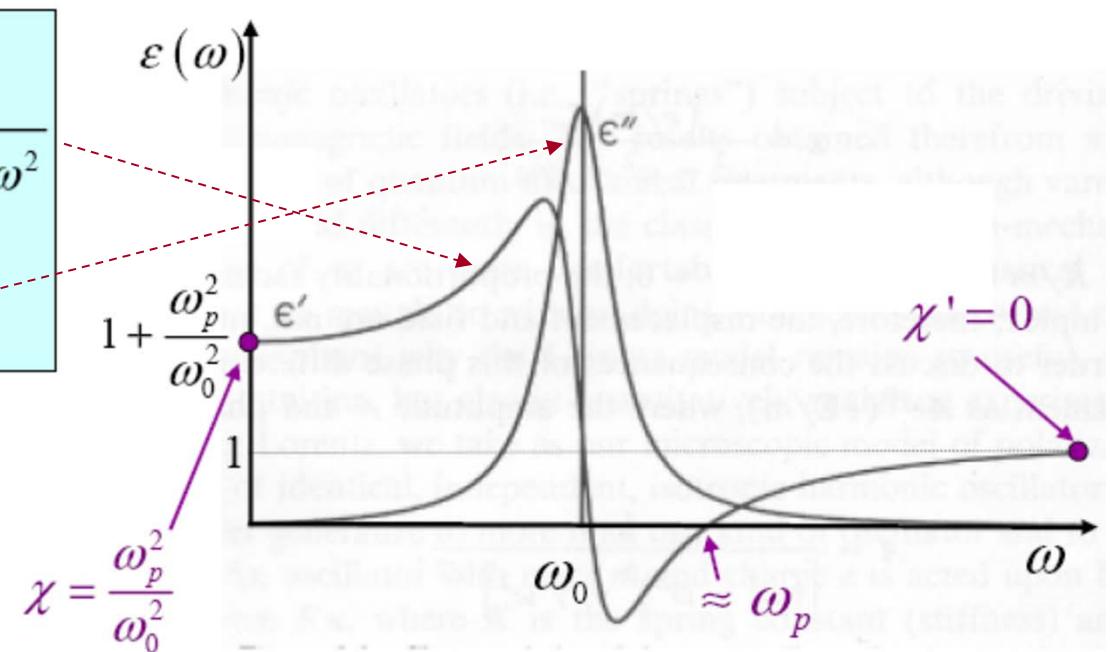
Frequency dependence ϵ

Relation of ϵ to χ : $\epsilon = 1 + \chi = 1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2 - i\gamma\omega}$

$\Rightarrow \epsilon = \epsilon' + i\epsilon'' = 1 + \chi' + i\chi'' = 1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2 - i\gamma\omega}$

$$\epsilon' = \chi'(\omega) = 1 + \frac{\omega_p^2(\omega_0^2 - \omega^2)}{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}$$

$$\epsilon'' = \chi''(\omega) = \frac{\omega_p^2\gamma\omega}{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}$$



Relation between n and ϵ

$$n = \sqrt{\epsilon} \quad \rightarrow$$

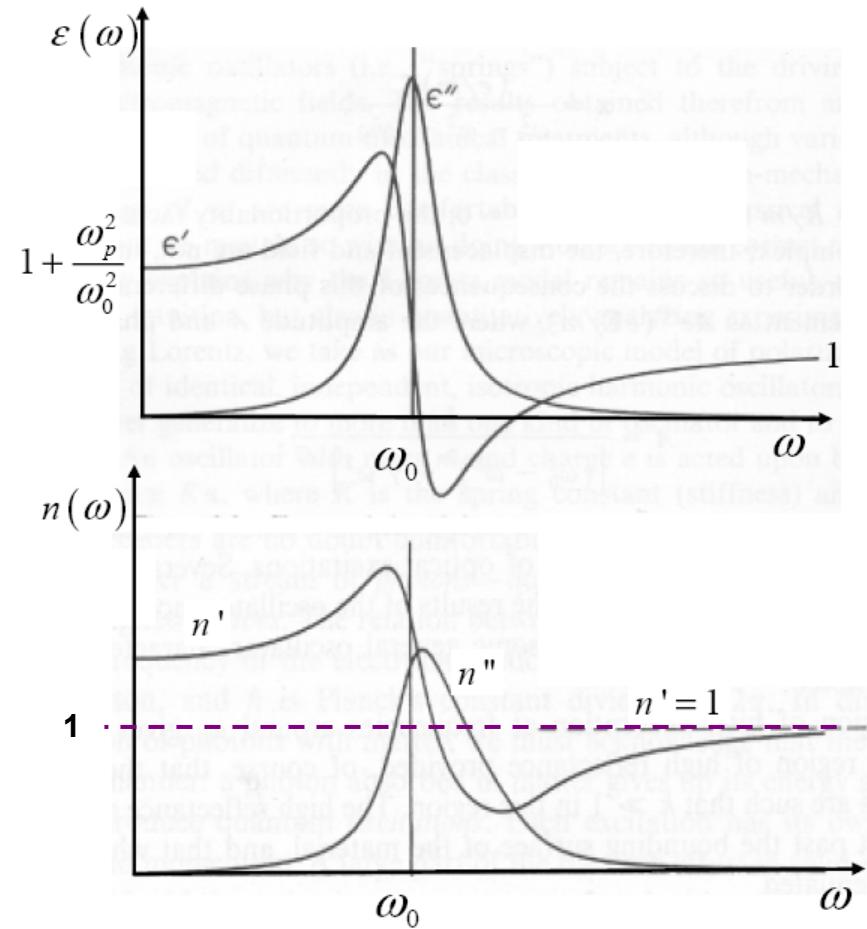
$$\begin{aligned}\epsilon_r' &= (n')^2 - (n'')^2 \\ \epsilon_r'' &= 2n'n''\end{aligned}$$

$$n' + in'' = \sqrt{\epsilon_r' + i\epsilon_r''}$$

$$\begin{aligned}n' &= \sqrt{\frac{(\epsilon_r')^2 + (\epsilon_r'')^2}{2} + \epsilon_r'} \\ n'' &= \sqrt{\frac{(\epsilon_r')^2 + (\epsilon_r'')^2}{2} - \epsilon_r'}\end{aligned}$$

- $\omega \ll \omega_0$: High n' \rightarrow low $v_{ph} = c/n'$
- $\omega \approx \omega_0$: Strong ω dependence v_{ph}
Large absorption ($\sim n''$)
- $\omega \gg \omega_0$: $n' = 1 \rightarrow v_{ph} = c$

Lorentz model (harmonic oscillator model)



Realistic Rarefied Media

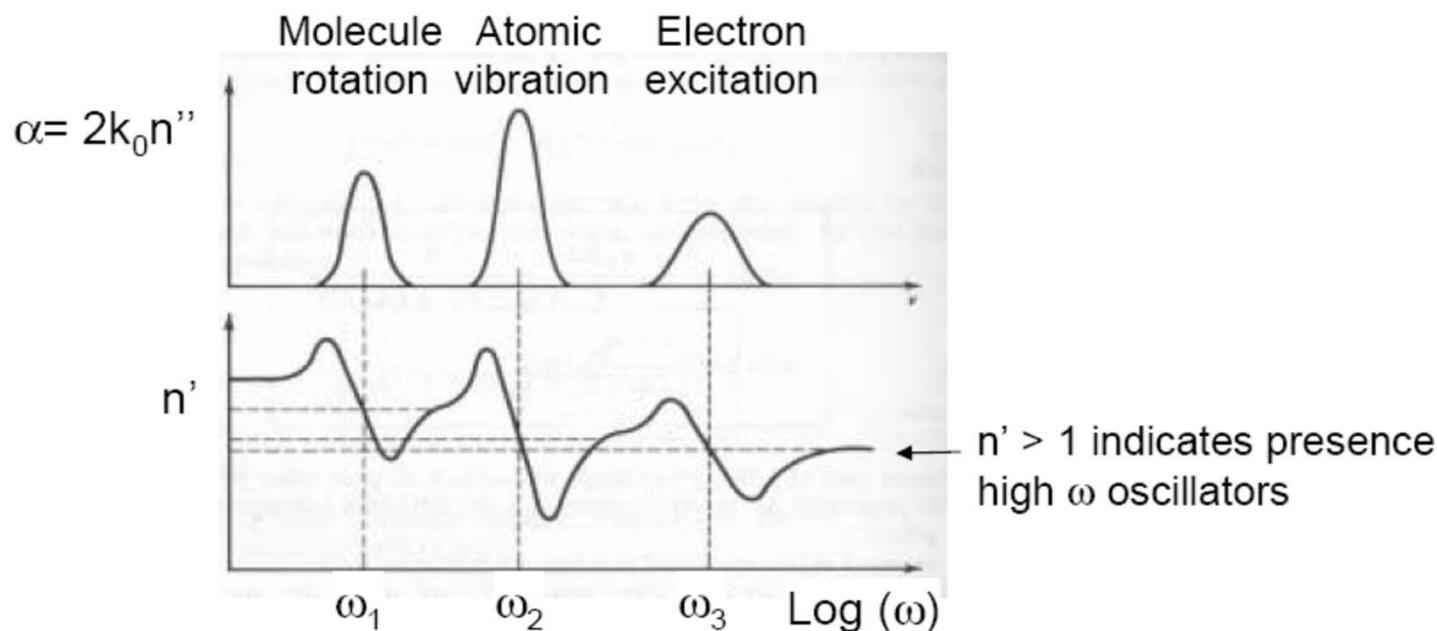
Realistic atoms have many resonances

- Resonances occur due to motion of the atoms (low ω) and electrons (high ω)

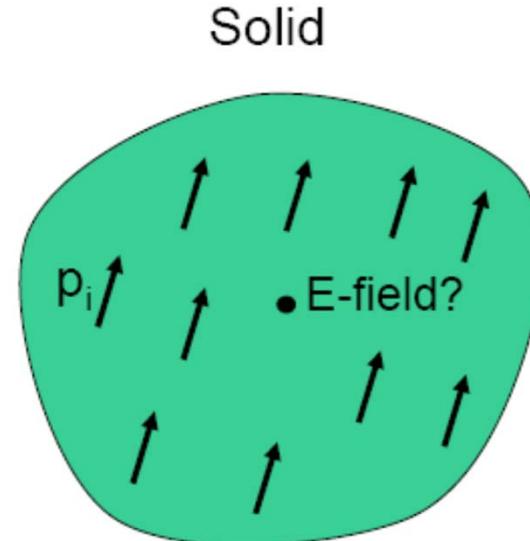
$$\Rightarrow \chi = \sum_k \frac{N_k e^2}{\epsilon_0 m} \frac{1}{\omega_k^2 - \omega^2 - i\gamma\omega}$$

Where N_k is the density of the electrons/atoms with a resonance at ω_k

Example of a realistic dependence of n' and n''



Solids : Atomic Polarizability (α) and χ



- Atom “feels” field from:
 - 1) Incident light beam
 - 2) Induced dipolar field from other atoms, p_i
- Local field:
$$\mathbf{E}_L = \mathbf{E}_0 + \mathbf{E}_I$$

Local field Induced dipolar field from all the other atoms
Field without matter

Electric Susceptibility of a Solid

Local field at an atom

- Local field: $\mathbf{E}_L = \mathbf{E}_0 + \mathbf{E}_I$
- Local field Field without matter All the other atoms

Induced dipolar field

- Example: For cubic symmetry: $\mathbf{E}_I = \frac{\mathbf{P}}{3\epsilon_0}$ (Solid state Phys. Books, e.g. Kittel, Ashcroft..)
- $\rightarrow \mathbf{E}_L = \mathbf{E}_0 + \frac{\mathbf{P}}{3\epsilon_0}$ (Similar relations can be derived for any solid)

유도해보시오!

Polarization of a solid

- Solid consists of atom type j at a concentration N_j

$$\mathbf{P} = \epsilon_0 \sum_j N_j \alpha_j \mathbf{E}_L = \epsilon_0 \sum_j N_j \alpha_j \left(\mathbf{E}_0 + \frac{\mathbf{P}}{3\epsilon_0} \right) = \epsilon_0 \sum_j N_j \alpha_j \mathbf{E}_0 + \sum_j N_j \alpha_j \frac{\mathbf{P}}{3}$$

$\underbrace{N_j \alpha_j}_{P_j}$

$$\rightarrow \mathbf{P} \left(1 - \frac{1}{3} \sum_j N_j \alpha_j \right) = \epsilon_0 \sum_j N_j \alpha_j \mathbf{E}_0 \rightarrow$$

$$\chi = \frac{P}{\epsilon_0 E} = \frac{\sum_j N_j \alpha_j}{1 - \frac{1}{3} \sum_j N_j \alpha_j}$$

Clausius-Mossotti Relation

Polarization of a solid

- Susceptibility: 

$$\chi = \frac{P}{\epsilon_0 E} = \frac{\sum_j N_j \alpha_j}{1 - \frac{1}{3} \sum_j N_j \alpha_j} \quad \text{I}$$

- Limit of low atomic concentration:
....or weak polarizability:
pretty good for gasses and glasses

$$\chi \approx \sum_j N_j \alpha_j \quad \text{II}$$

Clausius-Mossotti

- By definition: $\epsilon = 1 + \chi$
- Rearranging I gives

$$\frac{\epsilon - 1}{\epsilon + 2} = \frac{1}{3\epsilon_0} \sum_j N_j \alpha_j \quad \text{III}$$

- Conclusion: Dielectric properties of solids related to atomic polarizability
- This is very general!!

→ The Polarizability of a solid with volume V given by the Clausius-Mossotti relation is

$$\begin{aligned} \alpha_{solid} &= V \left(\sum_j N_j \alpha_j \right) = V \left[3\epsilon_0 \left(\frac{\epsilon_r - \epsilon_r^{embed}}{\epsilon_r + 2\epsilon_r^{embed}} \right) \right] = V \left[3 \left(\frac{\epsilon - \epsilon^{embed}}{\epsilon + 2\epsilon^{embed}} \right) \right] \\ &= \left(4\pi a^3 \right) \left(\frac{\epsilon - \epsilon^{embed}}{\epsilon + 2\epsilon^{embed}} \right) \text{ if the medium is a sphere with radius } a. \end{aligned}$$

For a simple case when $C_j = C$ for all atoms in solid,

$$\chi = \frac{\sum_j N_j \alpha_j}{1 - \frac{1}{3} \sum_j N_j \alpha_j} \quad \text{where} \quad \alpha_j = \left(\frac{e^2}{m \epsilon_0} \right) \frac{1}{\omega_j^2 - \omega^2 - i\gamma\omega} \quad \text{and} \quad \omega_j^2 = \frac{C_j}{m} = \frac{C}{m}$$

$$\chi = \frac{N\alpha}{1 - \frac{1}{3}N\alpha} = \left(\frac{Ne^2}{m \epsilon_0} \right) \frac{1}{\left(\frac{C}{m} - \frac{Ne^2}{3m \epsilon_0} \right) - \omega^2 - i\gamma\omega}$$

$$\chi(\omega) = \frac{\omega_p^2}{\omega_0^2 - \omega^2 - i\gamma\omega}, \text{ where } \omega_0^2 \equiv \frac{C}{m} - \frac{Ne^2}{3m \epsilon_0}$$

This is the same form as the single atomic gasses, except the different definition of ω_0 .

In general when C_j are not identical,

$$\chi(\omega) = \sum_j \frac{\omega_p^2 f_j}{(\omega_j^2 - \omega^2 - i\omega\gamma_j)}$$

f_j is the oscillation strength (the fraction of dipoles having ω_j)

B. $\sigma(\omega)$ in free-electron region: Drude model = Free-electron model

- Drude model : Lorenz model (Harmonic oscillator model) *without restoration force* (that is, free electrons which are not bound to a particular nucleus)

The equation of motion of a free electron (not bound to a particular nucleus; $C = 0$),

$$m \frac{d^2 \vec{r}}{dt^2} = -C\vec{r} - \frac{m}{\tau} \frac{d\vec{r}}{dt} - e\vec{E} \quad \stackrel{\substack{\text{If} \\ C=0}}{\Rightarrow} \quad m \frac{d\vec{v}}{dt} + m\gamma\vec{v} = -e\vec{E} \quad (\tau = \frac{1}{\gamma} : \text{relaxation time } \approx 10^{-14} \text{ s})$$

*Lorentz model
(Harmonic oscillator model)*

*Drude model
(free-electron model)*

The current density is defined : $\vec{J} = -N e \vec{v}$ $\left[\frac{A}{m^2} = \frac{C}{s \cdot m^2} \right]$

$$m \frac{d\vec{v}}{dt} + m\gamma\vec{v} = -e\vec{E} \quad \longrightarrow \quad \frac{d\vec{J}}{dt} + \gamma \vec{J} = \left(\frac{N e^2}{m} \right) \vec{E}$$

$$\frac{d\vec{J}}{dt} + \gamma \vec{J} = \left(\frac{Ne^2}{m_e} \right) \vec{E}$$

Assume that the applied electric field and the conduction current density are given by:

$$\vec{E}(\vec{r}) = \vec{E}_0(\vec{r}) \exp(-i\omega t) \longrightarrow \vec{J}(\vec{r}) = \vec{J}_0(\vec{r}) \exp(-i\omega t) \xleftarrow{\text{Local approximation to the current-field relation}}$$

Substituting into the equation of motion we obtain:

$$\begin{aligned} \frac{d[\vec{J}_0 \exp(-i\omega t)]}{dt} + \gamma \vec{J}_0 \exp(-i\omega t) &= -i\omega \vec{J}_0 \exp(-i\omega t) + \gamma \vec{J}_0 \exp(-i\omega t) \\ &= \left(\frac{Ne^2}{m_e} \right) \vec{E}_0 \exp(-i\omega t) \end{aligned}$$

Multiplying through by $\exp(+i\omega t)$:

$$(-i\omega + \gamma) \vec{J}_0 = \left(\frac{Ne^2}{m_e} \right) \vec{E}_0, \quad \text{or equivalently} \quad \longrightarrow \boxed{(-i\omega + \gamma) \vec{J} = \left(\frac{Ne^2}{m_e} \right) \vec{E}}$$

B. Drude model

$$(-i\omega + \gamma) \vec{J} = \left(\frac{Ne^2}{m_e} \right) \vec{E}$$

For static fields ($\omega=0$), $\vec{J} = \left(\frac{Ne^2}{m\gamma} \right) \vec{E} = \sigma_0 \vec{E}$, where $\sigma_0 = \frac{Ne^2}{m\gamma}$: static conductivity

For the general case of an oscillating applied field :

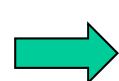
$$\vec{J} = \left[\frac{\sigma_0}{1 - (i\omega/\gamma)} \right] \vec{E} = \sigma(\omega) \vec{E},$$

$$\sigma(\omega) = \frac{\sigma_0}{1 - (i\omega/\gamma)} = \frac{Ne^2/m}{\gamma - i\omega} : \text{dynamic conductivity}$$

$$\varepsilon_r(\omega) = 1 + i \frac{\sigma(\omega)}{\varepsilon_0 \omega}$$

$$\varepsilon_r(\omega) = 1 + i \left\{ \frac{\sigma_0 c^2 \mu_0}{\omega [1 - (i\omega/\gamma)]} \right\} = 1 + i \frac{i\gamma}{i\gamma} \left\{ \frac{\sigma_0 c^2 \mu_0}{\omega [1 - (i\omega/\gamma)]} \right\} = 1 - \frac{\gamma \sigma_0 c^2 \mu_0}{\omega^2 + i\omega\gamma}$$

The plasma frequency is defined, $\omega_p^2 = \gamma \sigma_0 c^2 \mu_0 = \gamma \left(\frac{Ne^2}{m\gamma} \right) c^2 \mu_0 = \frac{Ne^2}{m\varepsilon_0}$



$$\varepsilon_r(\omega) = 1 - \frac{\omega_p^2}{\omega^2 + i\omega\gamma}, \quad \omega_p^2 = \frac{Ne^2}{m\varepsilon_0}$$

Note : Dynamic conductivity $\sigma(\omega)$

$$\varepsilon_r(\omega) = 1 + i \frac{\sigma(\omega)}{\varepsilon_0 \omega}$$

$$\sigma(\omega) = \frac{\sigma_0}{1 - (i\omega/\gamma)}$$

For very low frequencies, $(\omega/\gamma) \ll 1$,

$\sigma(\omega) \sim \sigma_0$: purely real and the electrons follow the electric field.

As the frequency of the applied field increases,

the inertia of electrons introduces a phase lag in the electron response to the field,
and $\sigma(\omega)$ is complex.

For very high frequencies, $(\omega/\gamma) \gg 1$,

$\sigma(\omega) \sim i\sigma_0\gamma/\omega$: purely imaginary, $\vec{J} \approx i (\sigma_0\gamma/\omega) \vec{E} = \left(e^{i\pi/2}\right) (\sigma_0\gamma/\omega) \vec{E}$

and the electron oscillations are 90° out of phase with the applied field.

Dielectric constant of free-electron plasma (Drude model)



$$\varepsilon_r(\omega) = 1 - \frac{\omega_p^2}{\omega^2 + i\omega\gamma} = \left(1 - \frac{\omega_p^2}{\omega^2 + \gamma^2}\right) + i\left(\frac{\omega_p^2\gamma}{\omega^3 + \omega\gamma^2}\right)$$

	Au	Ag	Cu
N/m^{-3}	$5.90 \cdot 10^{28}$	$5.76 \cdot 10^{28}$	$8.45 \cdot 10^{28}$
$\sigma_0/(\Omega m)^{-1}$	$4.9 \cdot 10^7$	$6.6 \cdot 10^7$	$6.5 \cdot 10^7$
m/m_e	0.99	0.96	1.49
$\hbar\omega_p/eV$	9.1	9.1	8.8
$\tau/f\text{s}$	29	40	40

$$\omega_p^2 = \gamma \sigma_0 c^2 \mu_0 = \frac{N e^2}{m \epsilon_0}$$

$$\sigma_0 = \frac{N e^2}{m \gamma} : static conductivity$$

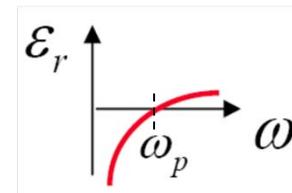
(1) For an optical frequency, $\omega_{\text{visible}} \gg \gamma$

$$\varepsilon_r(\omega) \approx \left(1 - \frac{\omega_p^2}{\omega^2}\right) + i\left(\frac{\omega_p^2}{\omega^3/\gamma}\right) = \left(1 - \frac{\omega_p^2}{\omega^2}\right) + i\left(\frac{\omega_p^2}{\omega^3\tau}\right)$$

(2) Ideal case : metal as an undamped free-electron gas

- no decay (infinite relaxation time)
- no interband transitions

$$\varepsilon_r(\omega) \xrightarrow[\gamma \rightarrow 0]{\tau \rightarrow \infty} \varepsilon_r(\omega) = 1 - \frac{\omega_p^2}{\omega^2}$$



Note : What is the actual meaning of ω_p

If the electrons in a plasma are displaced from a uniform background of ions, electric fields will be built up in such a direction as to restore the neutrality of the plasma by pulling the electrons back to their original positions.

Because of their inertia, the electrons will overshoot and oscillate around their equilibrium positions with a characteristic frequency known as the **plasma frequency**.

$E_s = \sigma_o / \epsilon_o = Ne(\delta x) / \epsilon_o$: electrostatic field by small charge separation δx

$\delta x = \delta x_o \exp(-i\omega_p t)$: small-amplitude oscillation

$$m \frac{d^2(\delta x)}{dt^2} = (-e)E_s \quad \Rightarrow \quad -m\omega_p^2 = -\frac{Ne^2}{\epsilon_o} \quad \Rightarrow \quad \omega_p^2 = \frac{Ne^2}{m\epsilon_o}$$

Note : plasma wavelength , λ_p

$$\lambda_p = \frac{2\pi c}{\omega_p}$$

Note : What happens in $\epsilon(\omega)$ when $\omega \sim \omega_p$?

$$\nabla \times \nabla \times E = \nabla(\nabla \cdot E) - \nabla^2 E = -\mu_0 \frac{\partial^2 D}{\partial t^2} \quad (\text{assume } J = 0)$$

$$k(k \cdot E) - k^2 E = -\epsilon(k, \omega) \frac{\omega^2}{c^2} E \quad : E = E_0 e^{j(k \cdot r - \omega t)}$$

(1) For transverse waves, $k \cdot E = 0 \longrightarrow k^2 = \epsilon(k, \omega) \frac{\omega^2}{c^2}$

(2) For longitudinal waves, $k(k \cdot E) - k^2 E = 0 \longrightarrow \boxed{\epsilon(k, \omega) = 0}$

Therefore, at the plasma frequency $\omega = \omega_p$,

$$\epsilon(k, \omega = \omega_p) = 0$$

→ means that at the plasma frequency only a longitudinal wave can propagate through.

$$D = 0 = \epsilon_0 E + P$$

→ E is a pure depolarization field (No transverse field strength in media).

→ The quanta of these longitudinal charge oscillations are called plasmons (or, volume plasmons)

→ Volume plasmons do not couple to transverse EM waves
(can be only excited by particle impact)

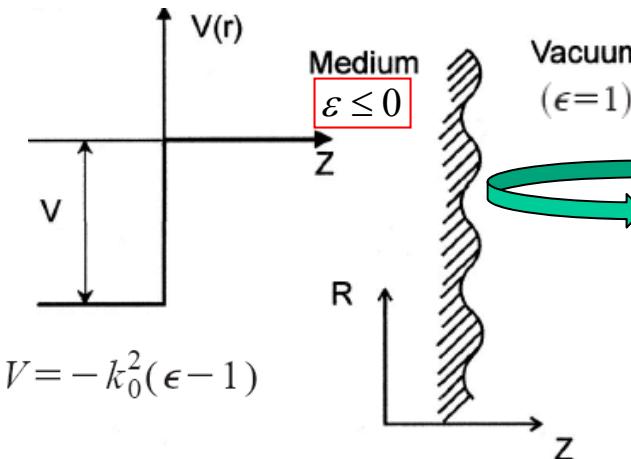
Note : What happens in $\epsilon(\omega)$ when $\omega \sim \omega_p$?

Optical potential

$$V = -k_0^2(\epsilon - 1)$$

Incident energy

$$K_{inc} = k_0^2$$



(N. Garcia, et. Al, "Zero permittivity materials", APL, 80, 1120 (2002))

→ If $\epsilon(\omega) \leq 0$, $K_{inc} \leq V$, the incident wave is completely reflected.

→ $\nabla \times \mathbf{H} = 0$, $\nabla \times \mathbf{E} = i\omega\mu_0\mathbf{H}$ $\nabla^2\mathbf{E} = 0$,

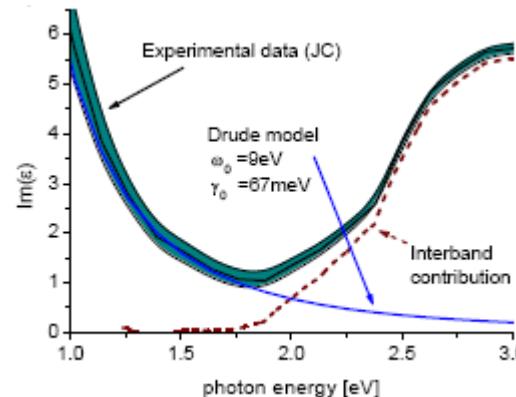
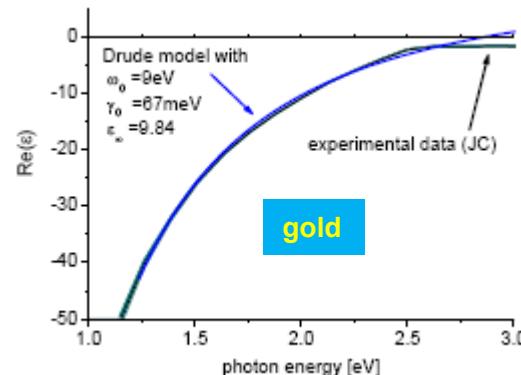
→ Wave propagation in this material can happen *only with phase velocity being infinitely large* satisfying the “static-like” equation

(A. Mario, et. Al, "Epsilon-near-zero metamaterials and electromagnetic sources: Tailoring the radiation phase pattern", PR B, 75, 155410, 2007)

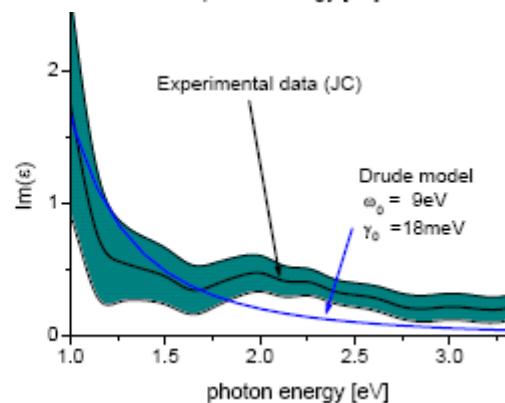
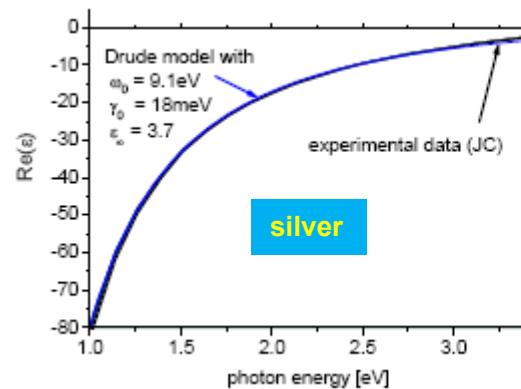
C. $\chi_b(\omega)$ in bound-electron region : Drude-Sommerfeld model = Quasi-free-electron model

$$\varepsilon_r(\omega) = \varepsilon_\infty + i \frac{\sigma(\omega)}{\varepsilon_0 \omega} \quad [\varepsilon_\infty = 1 + \chi_b(\omega), \text{ usually } 1 \leq \varepsilon_\infty \leq 10]$$

$$\varepsilon_r(\omega) = \varepsilon_\infty - \frac{\omega_p^2}{\omega^2 + i\omega\gamma} = \left(\varepsilon_\infty - \frac{\omega_p^2}{\omega^2 + \gamma^2} \right) + i \left(\frac{\omega_p^2 \gamma}{\omega^3 + \omega\gamma^2} \right)$$



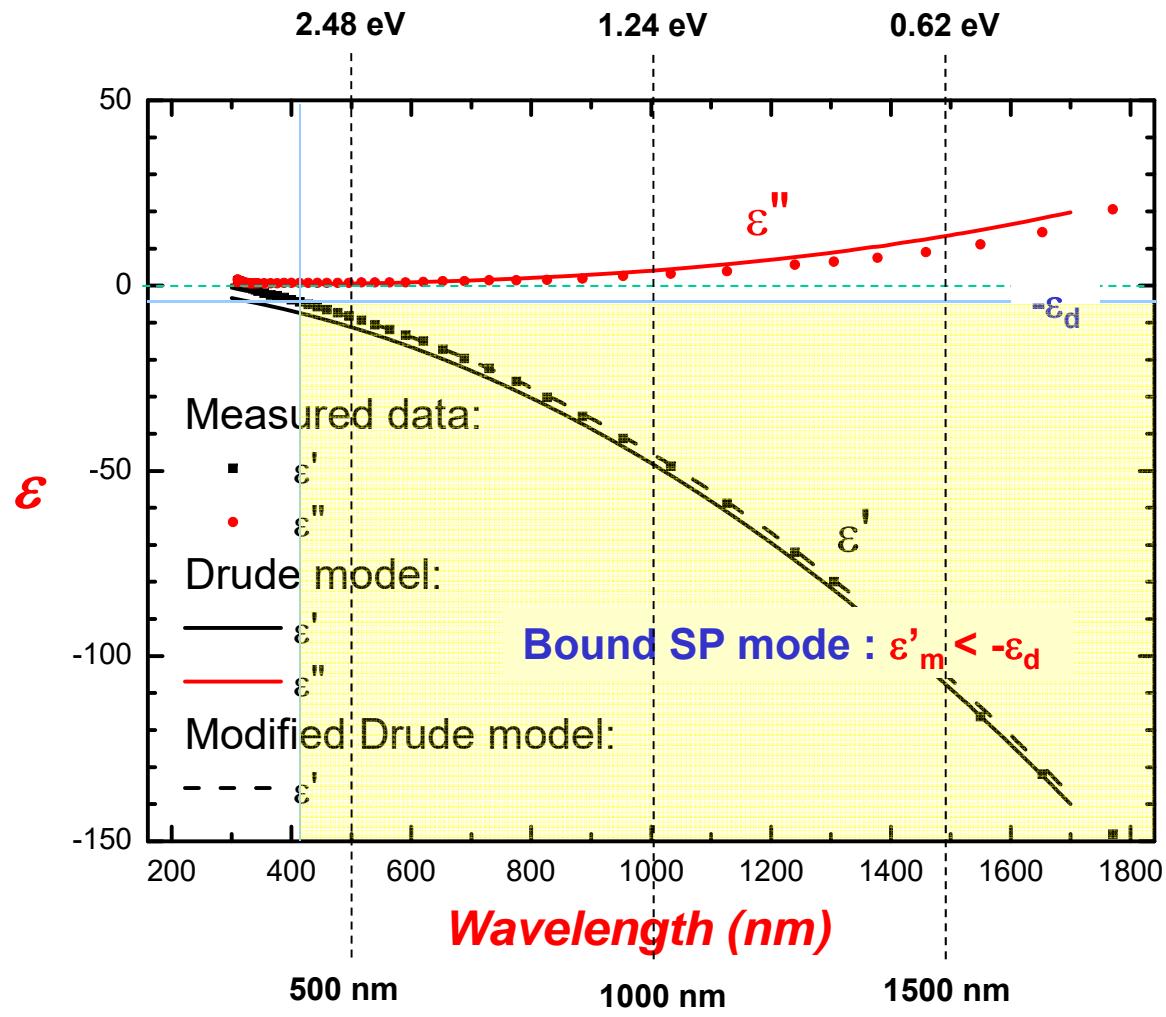
	Au	Ag	Au	Ag	Au	Ag
$\hbar\omega_p/\text{eV}$	9.0	9.1	8.4	8.75	8-10	8.3-9.2
τ/fs	10	36	6-19	8.6	4-26	15-23
ε_∞	9.84	3.7	6.5-7.5	4.45		
E_{IB}	1.75	1.75	1.5	1.5		
reference	this work		PW		Land./Börn.	



Tab. 2.2: Drude-parameters at optical frequencies for Au and Ag. Interband contributions below E_{IB} are taken into account by ε_∞ . Values in this work taken from fits in the region $1.4 < E/\text{eV} < 1.75$ of data reported by Johnson and Christy (1972) (cf. Fig. 2.1), PW: Winsemius (1972), Land./Börn.: collection of values by Foiles (1985).

Plasmons in metal nanostructures,
Dissertation, University of Munich
by Carsten Sonnichsen, 2001

Measured data and modified Drude model for Ag:



Drude model:

$$\epsilon' = 1 - \frac{\omega_p^2}{\omega^2}, \quad \epsilon'' = \frac{\omega_p^2}{\omega^3} \gamma$$

Modified Drude model:

$$\epsilon' = \epsilon_\infty - \frac{\omega_p^2}{\omega^2}, \quad \epsilon'' = \frac{\omega_p^2}{\omega^3} \gamma$$

Contribution of
bound electrons

For Ag, $\epsilon_\infty = 3.7$

$$\omega_p = 9.1 \text{ eV}$$

→ For Ag, good matched bellow 2.5 eV

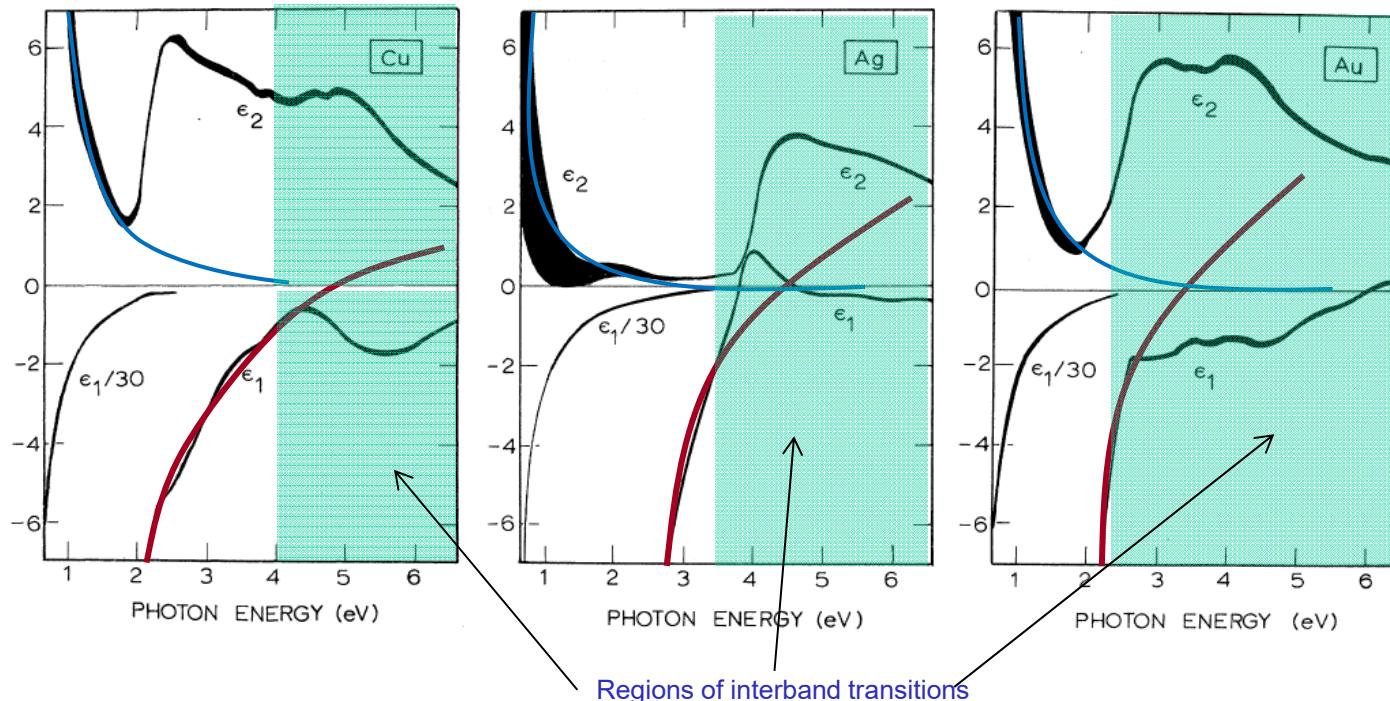
Let's compare the modified Drude model with experimental measurement

$$\epsilon_r(\omega) = \left(\epsilon_\infty - \frac{\omega_p^2}{\omega^2 + \gamma^2} \right) + i \left(\frac{\omega_p^2 \gamma}{\omega^3 + \omega \gamma^2} \right)$$

TABLE II. Optical masses and the relaxation times for copper, silver, and gold.

	m_0 (electron masses)	τ (sec)
Copper	1.49 ± 0.06	$(6.9 \pm 0.7) \times 10^{-15}$
Silver	0.96 ± 0.04	$(31 \pm 12) \times 10^{-15}$
Gold	0.99 ± 0.04	$(9.3 \pm 0.9) \times 10^{-15}$

P. B. Johnson and R. W. Christy, "Optical constants of the noble metals", Phys. Rev. B, 6, pp. 4370-4379 (1972).



→ Not good matched in interband regions! → Need something more.

P. B. Johnson and R. W. Christy, "Optical constants of the noble metals", Phys. Rev. B, 6, pp. 4370-4379 (1972).

TABLE I. Optical constants for copper, silver, and gold as well as the approximate errors in n and k .

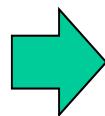
eV	Copper		Silver		Gold		Error	
	n	k	n	k	n	k	Δn	Δk
0.64	1.09	13.43	0.24	14.08	0.92	13.78	± 0.18	± 0.65
0.77	0.76	11.12	0.15	11.85	0.56	11.21	± 0.08	± 0.30
0.89	0.60	9.439	0.13	10.10	0.43	9.519	± 0.06	± 0.17
1.02	0.48	8.245	0.09	8.828	0.35	8.145	± 0.04	± 0.10
1.14	0.36	7.217	0.04	7.795	0.27	7.150	± 0.03	± 0.07
1.26	0.32	6.421	0.04	6.992	0.22	6.350	± 0.02	± 0.05
1.39	0.30	5.768	0.04	6.312	0.17	5.663	± 0.02	± 0.03
1.51	0.26	5.180	0.04	5.727	0.16	5.083	± 0.02	± 0.025
1.64	0.24	4.665	0.03	5.242	0.14	4.542	± 0.02	± 0.015
1.76	0.21	4.205	0.04	4.838	0.13	4.103	± 0.02	± 0.010
1.88	0.22	3.747	0.05	4.483	0.14	3.697	± 0.02	± 0.007
2.01	0.30	3.205	0.06	4.152	0.21	3.272	± 0.02	± 0.007
2.13	0.70	2.704	0.05	3.858	0.29	2.863	± 0.02	± 0.007
2.26	1.02	2.577	0.06	3.586	0.43	2.455	± 0.02	± 0.007
2.38	1.18	2.608	0.05	3.324	0.62	2.081	± 0.02	± 0.007
2.50	1.22	2.564	0.05	3.093	1.04	1.833	± 0.02	± 0.007
2.63	1.25	2.483	0.05	2.869	1.31	1.849	± 0.02	± 0.007
2.75	1.24	2.397	0.04	2.657	1.38	1.914	± 0.02	± 0.007
2.88	1.25	2.305	0.04	2.462	1.45	1.948	± 0.02	± 0.007
3.00	1.28	2.207	0.05	2.275	1.46	1.958	± 0.02	± 0.007
3.12	1.32	2.116	0.05	2.070	1.47	1.952	± 0.02	± 0.007
3.25	1.33	2.045	0.05	1.864	1.46	1.933	± 0.02	± 0.007
3.37	1.36	1.975	0.07	1.657	1.48	1.895	± 0.02	± 0.007
3.50	1.37	1.918	0.10	1.419	1.50	1.866	± 0.02	± 0.007
3.62	1.36	1.864	0.14	1.142	1.48	1.871	± 0.02	± 0.007
3.74	1.34	1.821	0.17	0.829	1.48	1.883	± 0.02	± 0.007
3.87	1.38	1.783	0.81	0.392	1.54	1.898	± 0.02	± 0.007
3.99	1.38	1.729	1.13	0.616	1.53	1.893	± 0.02	± 0.007
4.12	1.40	1.679	1.34	0.964	1.53	1.889	± 0.02	± 0.007
4.24	1.42	1.633	1.39	1.161	1.49	1.878	± 0.02	± 0.007
4.36	1.45	1.633	1.41	1.264	1.47	1.869	± 0.02	± 0.007
4.49	1.46	1.646	1.41	1.331	1.43	1.847	± 0.02	± 0.007
4.61	1.45	1.668	1.38	1.372	1.38	1.803	± 0.02	± 0.007
4.74	1.41	1.691	1.35	1.387	1.35	1.749	± 0.02	± 0.007
4.86	1.41	1.741	1.33	1.393	1.33	1.688	± 0.02	± 0.007
4.98	1.37	1.783	1.31	1.389	1.33	1.631	± 0.02	± 0.007
5.11	1.34	1.799	1.30	1.378	1.32	1.577	± 0.02	± 0.007
5.23	1.28	1.802	1.28	1.367	1.32	1.536	± 0.02	± 0.007
5.36	1.23	1.792	1.28	1.357	1.30	1.497	± 0.02	± 0.007
5.48	1.18	1.768	1.26	1.344	1.31	1.460	± 0.02	± 0.007
5.60	1.13	1.737	1.25	1.342	1.30	1.427	± 0.02	± 0.007
5.73	1.08	1.699	1.22	1.336	1.30	1.387	± 0.02	± 0.007
5.85	1.04	1.651	1.20	1.325	1.30	1.350	± 0.02	± 0.007
5.98	1.01	1.599	1.18	1.312	1.30	1.304	± 0.02	± 0.007
6.10	0.99	1.550	1.15	1.296	1.33	1.277	± 0.02	± 0.007
6.22	0.98	1.493	1.14	1.277	1.33	1.251	± 0.02	± 0.007
6.35	0.97	1.440	1.12	1.255	1.34	1.226	± 0.02	± 0.007
6.47	0.95	1.388	1.10	1.232	1.32	1.203	± 0.02	± 0.007
6.60	0.94	1.337	1.07	1.212	1.28	1.188	± 0.02	± 0.007

D. bound-electron region : Extended Drude (Drude-Lorentz) model

Alexandre Vial, et.al, "Improved analytical fit of gold dispersion: Application to the modeling of extinction spectra with a finite-difference time-domain method", Phys. Rev. B, 71, 085416 (2005).

The Drude-Lorentz model consists *in addition of one Lorentz term to the modified Drude model.*

Since the Lorentz terms of insulators have a general form of $\chi(\omega) = \sum_j \frac{f_j \omega_p^2}{\omega_j^2 - \omega^2 - i\omega\gamma_j}$



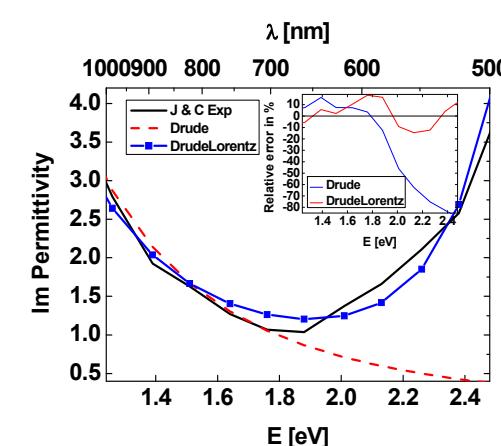
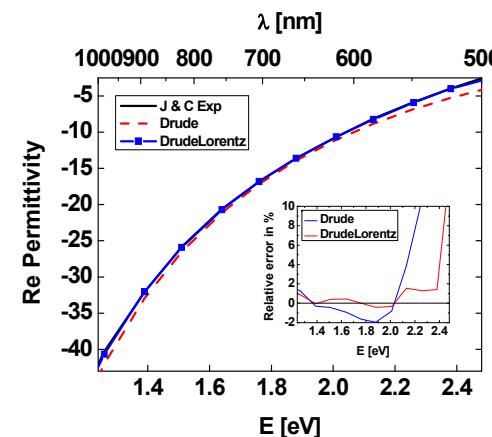
For gold (Au),

$$\varepsilon_r(\omega) = \varepsilon_\infty - \left(\frac{\omega_p^2}{\omega^2 + i\omega\gamma} \right) - \left(\frac{\Delta\epsilon \Omega_L^2}{(\omega^2 - \Omega_L^2) + i\omega\Gamma_L} \right)$$

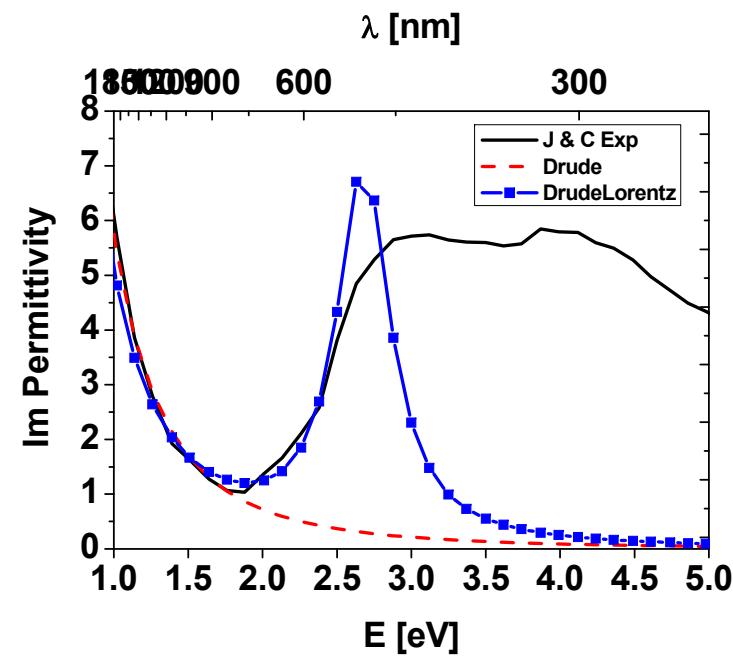
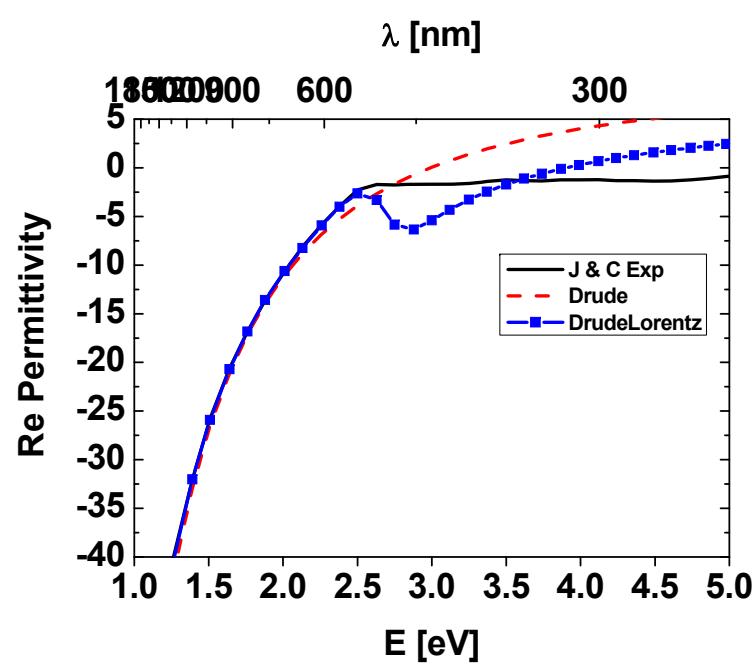
TABLE I. Values of the parameters used for the optimization of the Drude and the Drude-Lorentz models. The value of the fitness function is given in the last column.

	ϵ_∞	$\omega_D/2\pi$ (THz)	$\gamma_D/2\pi$ (THz)	$\Omega_L/2\pi$ (THz)	$\Gamma_L/2\pi$ (THz)	$\Delta\epsilon$	Φ
Drude	9.0685	2155.6	18.36				431.46
Drude-Lorentz	5.9673	2113.6	15.92	650.07	104.86	1.09	14.521

Excellent agreement
within 500 nm ~ 1 μm
for gold (Au)



But, not correct at higher energy



In summary :

Lorentz model for dielectric (insulator)

$$\varepsilon_r(\omega) = 1 + \frac{\sum_j N_j \alpha_j}{1 - \frac{1}{3} \sum_j N_j \alpha_j}, \quad \alpha_j = \frac{\omega_p^2 / N_j}{\omega_j^2 - \omega^2 - i\gamma\omega}, \quad \omega_j^2 = \frac{C_j}{m}$$

Drude model for metal in free-electron region

$$\varepsilon_r(\omega) = 1 - \frac{\omega_p^2}{\omega^2 + i\omega\gamma} = \left(1 - \frac{\omega_p^2}{\omega^2 + \gamma^2} \right) + i \left(\frac{\omega_p^2 \gamma}{\omega^3 + \omega\gamma^2} \right)$$

Modified Drude model for metal in bound-electron region

$$\varepsilon_r(\omega) = \varepsilon_\infty - \frac{\omega_p^2}{\omega^2 + i\omega\gamma} = \left(\varepsilon_\infty - \frac{\omega_p^2}{\omega^2 + \gamma^2} \right) + i \left(\frac{\omega_p^2 \gamma}{\omega^3 + \omega\gamma^2} \right)$$

Extended Drude (Drude-Lorentz) model

$$\varepsilon_r(\omega) = \varepsilon_\infty - \left(\frac{\omega_p^2}{\omega^2 + i\omega\gamma} \right) - \left(\frac{\Delta\varepsilon \Omega_L^2}{(\omega^2 - \Omega_L^2) + i\omega \Gamma_L} \right)$$

Also, keep in mind.....

n' and n'' vs χ' and χ'' vs ϵ' and ϵ''

All pairs (n' and n'', χ' and χ'' , ϵ' and ϵ'') describe the same physics

For some problems one set is preferable for others another

n' and n'' used when discussing wave propagation

$$E(z,t) = \text{Re} \left\{ E(z,\omega) \exp \left(-i\beta z - \frac{\alpha}{2}z + i\omega t \right) \right\} \quad \text{where} \quad \begin{array}{ll} \beta = k_0 n' & \text{Phase propagation} \\ \alpha = -2k_0 n'' & \text{absorption} \end{array}$$

χ' and χ''
 ϵ' and ϵ'' } used when discussing microscopic origin of optical effects

Inter relationships

Example: n and ϵ

From $n = \sqrt{\epsilon_r}$ \rightarrow
 $n' + in'' = \sqrt{\epsilon_r' + i\epsilon_r''}$

$$\begin{aligned}\epsilon_r' &= (n')^2 - (n'')^2 \\ \epsilon_r'' &= 2n'n''\end{aligned}$$

and

$$\begin{aligned}n' &= \sqrt{\frac{(\epsilon_r')^2 + (\epsilon_r'')^2 + \epsilon_r'}{2}} \\ n'' &= \sqrt{\frac{(\epsilon_r')^2 + (\epsilon_r'')^2 - \epsilon_r'}{2}}\end{aligned}$$