5. Superposition of Waves

Last Lecture

• Wave Equations
• Harmonic Waves
• Plane Waves
• EM Waves

This Lecture

• Superposition of Waves
Superposition Principle

\[ \psi_1(\mathbf{r}, t), \quad I_1 \]

\[ \psi_2(\mathbf{r}, t), \quad I_2 \]

\[ \psi(\mathbf{r}, t) = \psi_1(\mathbf{r}, t) + \psi_2(\mathbf{r}, t) \]

or \[ I = I_1 + I_2 \]

or Both?

Formally speaking,

If \( \psi_1 \) and \( \psi_2 \) are independently solutions of the wave equation,

\[ \nabla^2 \psi = \frac{1}{v^2} \frac{\partial^2 \psi}{\partial t^2} \]

then the linear combination,

\[ \psi = a\psi_1 + b\psi_2 \]

\( a, b : \) constants

is also a solution.

For electromagnetic waves,

\[ \mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 \]
\[ \mathbf{H} = \mathbf{H}_1 + \mathbf{H}_2 \]

(Orientation of the fields must be taken into account.)
Suppose that $\psi_1$ and $\psi_2$ are both solutions of the wave equation:

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = \nabla^2 \psi = \frac{1}{v^2} \frac{\partial^2 \psi}{\partial t^2}$$

Then any linear combination of $\psi_1$ and $\psi_2$ is also a solution of the wave equation. For example, if $a$ and $b$ are constants, then

$$\psi = a \psi_1 + b \psi_2$$

is also a solution of the wave equation.
5-2. Superposition of Waves of the Same Frequency

\[ E_R = E_1 + E_2 \]

\[ = E_{01} \cos(\alpha_1 - \omega t) + E_{02} \cos(\alpha_2 - \omega t) \]
Constructive interference and Destructive interference

Constructive interference:

\[ \alpha_2 - \alpha_1 = (2\pi) m \]

\[ E_1 + E_2 = E_{01} \cos(\alpha_1 - \omega t) + E_{02} \cos(\alpha_1 + 2\pi m - \omega t) \]

\[ = (E_{01} + E_{02}) \cos(\alpha_1 - \omega t) \]

Destructive interference:

\[ \alpha_2 - \alpha_1 = (2m + 1) \pi \]

\[ E_1 + E_2 = E_{01} \cos(\alpha_1 - \omega t) + E_{02} \cos(\alpha_1 + (2m + 1) \pi - \omega t) \]

\[ = (E_{01} - E_{02}) \cos(\alpha_1 - \omega t) \]
General superposition of the Same Frequency

\[ E_R = \text{Re} \left( E_{01} e^{i(\alpha_1 - \omega t)} + E_{02} e^{i(\alpha_2 - \omega t)} \right) \]

Defining \[ E_0 e^{i\alpha} \equiv E_{01} e^{i\alpha_1} + E_{02} e^{i\alpha_2} \]

\[ \Rightarrow E_R = \text{Re} \left( E_0 e^{i(\alpha - \omega t)} \right) = E_0 \cos(\alpha - \omega t) \]

\[ E_0 = \sqrt{E_{01}^2 + E_{02}^2 + 2E_{01}E_{02} \cos(\alpha_2 - \alpha_1)} \]

\[ \tan \alpha = \frac{E_{01} \sin \alpha_1 + E_{02} \sin \alpha_2}{E_{01} \cos \alpha_1 + E_{02} \cos \alpha_2} \]
The relations just developed can be extended to the addition of an arbitrary number $N$ of waves:

$$E(x, t) = E_0 \cos(\alpha - \omega t) = \sum_{i=1}^{N} E_{0i} \cos(\alpha_i - \omega t)$$

$$E_0^2 = \sum_{i=1}^{N} E_{0i}^2 + \sum_{j>i}^{N} \sum_{i=1}^{N} 2E_{0i} E_{0j} \cos(\alpha_j - \alpha_i)$$

$$\tan \alpha = \frac{\sum_{i=1}^{N} E_{0i} \sin \alpha_i}{\sum_{i=1}^{N} E_{0i} \cos \alpha_i}$$
5-3. Random and coherent sources

\[ E_0^2 = \sum_{i=1}^{N} E_{0i}^2 + \sum_{j>i}^{N} \sum_{i=1}^{N} 2E_{0i} E_{0j} \cos(\alpha_j - \alpha_i) \]

Randomly phased sources:

\[ E_0^2 = \sum_{i=1}^{N} E_{0i}^2 + \sum_{j>i}^{N} \sum_{i=1}^{N} 2E_{0i} E_{0j} \cos(\alpha_j - \alpha_i) = \sum_{i=1}^{N} E_{0i}^2 \quad \rightarrow \quad I_0 = \sum_{i=1}^{N} I_{0i} \]

The resultant irradiance (intensity) of the randomly phased sources is the sum of the individual irradiances.

Coherent and in-phase sources:

\[ E_0^2 = \sum_{i=1}^{N} E_{0i}^2 + \sum_{j>i}^{N} \sum_{i=1}^{N} 2E_{0i} E_{0j} \cos(\alpha_j - \alpha_i) = \sum_{i=1}^{N} E_{0i}^2 + \sum_{j>i}^{N} \sum_{i=1}^{N} 2E_{0i} E_{0j} \]

\[ I_0 = \left( \sum_{i=1}^{N} E_{0i} \right)^2 \quad \rightarrow \quad I_0 = N^2 \times I_{01} \]

The resultant irradiance (intensity) of the coherent sources is \( N^2 \) times the individual irradiances.
5-4. Standing waves

\[ E_R = E_1 + E_2 = E_0 \sin(\omega t + kx) + E_0 \sin(\omega t - kx - \varphi_R) \]

\[ E_R = 2E_0 \cos(kx + \frac{\varphi_R}{2}) \sin(\omega t - \frac{\varphi_R}{2}) \]

\[ \{\varphi_R = \pi\} \rightarrow E_R = (2E_0 \sin kx) \cos \omega t \quad kx = \frac{2\pi x}{\lambda} = m\pi \]

Nodes at \( x = m\left(\frac{\lambda}{2}\right) \)

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In a laser cavity

\[ d = m \left( \frac{\lambda_m}{2} \right) \]

\[ \lambda_m = \left( \frac{2d}{m} \right) \quad \nu_m = \frac{c}{\lambda_m} = m \left( \frac{c}{2d} \right) \]
5-5. The beat phenomenon

\[ \psi_1(x) = A \cos(k_1 x) \quad \text{and} \quad \psi_2(x) = A \cos(k_2 x) \]

\[ \psi(x) = \psi_1(x) + \psi_2(x) = A \left( \cos(k_1 x) + \cos(k_2 x) \right) \]

\[ \text{cos} \theta_1 + \cos \theta_2 = 2 \cos \left( \frac{\theta_1 + \theta_2}{2} \right) \cos \left( \frac{\theta_1 - \theta_2}{2} \right) \]

\[ = 2A \cos \left( \frac{k_1 - k_2}{2} x \right) \cos \left( \frac{k_1 + k_2}{2} x \right) \]

\[ \psi(x) = 2A \left( \cos k_g x \right) \left( \cos k_p x \right) \]

\[ k_g = \frac{k_1 - k_2}{2} \]

\[ k_p = \frac{k_1 + k_2}{2} \]
Superposition of Waves with Different Frequency

If the 2 waves have approximately equal wavelengths, \( \lambda_1 \approx \lambda_2 \) \( \Rightarrow \Delta \lambda = \lambda_2 - \lambda_1 \ll \lambda_1, \lambda_2 \)

\[
\begin{align*}
  k_p &= \frac{k_1 + k_2}{2} = \frac{2\pi}{2} \left( \frac{1}{\lambda_1} + \frac{1}{\lambda_2} \right) = \frac{2\pi}{2} \left( \frac{\lambda_1 + \lambda_2}{\lambda_1 \lambda_2} \right) = \frac{2\pi}{2} \frac{2\lambda_1}{\lambda_1^2} = \frac{2\pi}{\lambda_1} \\
  k_g &= \frac{k_1 - k_2}{2} = \frac{2\pi}{2} \left( \frac{1}{\lambda_1} - \frac{1}{\lambda_2} \right) = \frac{2\pi}{2} \left( \frac{\lambda_2 - \lambda_1}{\lambda_1 \lambda_2} \right) = \frac{2\pi}{2} \frac{\Delta \lambda}{\lambda_1^2} = \frac{2\pi}{\lambda_1} \frac{\Delta \lambda}{2\lambda_1} = \frac{2\pi}{\Lambda} \\
  \Lambda &= \left( \frac{2\lambda_1}{\Delta \lambda} \right) \lambda_1 \quad \text{: beat wavelength}
\end{align*}
\]

\[
\psi(x) \equiv 2A \cos\left( \frac{2\pi}{\Lambda} x \right) \cos\left( \frac{2\pi}{\lambda_1} x \right)
\]
Phase velocity and Group velocity

When the waves have also a time dependence,

\[ \psi_1(x,t) = A \cos(k_1 x - \omega_1 t) \]
\[ \psi_2(x,t) = A \cos(k_2 x - \omega_2 t) \]

\[ k_p = \frac{k_1 + k_2}{2} \]
\[ \omega_p = \frac{\omega_1 + \omega_2}{2} \quad \text{higher frequency wave} \]

\[ k_g = \frac{k_1 - k_2}{2} \]
\[ \omega_g = \frac{\omega_1 - \omega_2}{2} \quad \text{lower frequency wave (envelope)} \]

Phase velocity:

\[ v_p = \frac{\omega_p}{k_p} = \frac{\omega_1 + \omega_2}{k_1 + k_2} \approx \frac{\omega}{k} \]

Group velocity:

\[ v_g = \frac{\omega_g}{k_g} = \frac{\omega_1 - \omega_2}{k_1 - k_2} \approx \frac{d\omega}{dk} \]

\[ v_g = \frac{d\omega}{dk} = \frac{d}{dk}(k v_p) = v_p + k \left( \frac{dv_p}{dk} \right) \]
\[ = v_p + k \frac{d}{dk} \left( \frac{c}{n} \right) = v_p + k \left( \frac{-c}{n^2} \right) \left( \frac{dn}{dk} \right) = v_p \left[ 1 - \frac{k}{n} \left( \frac{dn}{dk} \right) \right] \]
\[ = v_p \left[ 1 + \frac{\lambda}{n} \left( \frac{dn}{d\lambda} \right) \right] \quad \left( k = 2\pi / \lambda \right) \]
Phase velocity and Group velocity

Phase velocity:
\[ v_p = \frac{\omega_p}{k_p} = \frac{\omega_1 + \omega_2}{k_1 + k_2} \approx \frac{\omega}{k} \]

Group velocity:
\[ v_g = \frac{\omega_g}{k_g} = \frac{\omega_1 - \omega_2}{k_1 - k_2} \approx \frac{d\omega}{dk} \]