Chapter 7. Interference of Light

Last Lecture
• Superposition of waves
• Laser

This Lecture
• Two-Beam Interference
• Young’s Double Slit Experiment
• Virtual Sources
• Newton’s Rings
• Film Thickness Measurement by Interference
• Stokes Relations
• Multiple-Beam Interference in a Parallel Plate
Consider two waves \( \vec{E}_1 \) and \( \vec{E}_2 \) that have the same frequency \( \omega \):

\[
\vec{E}_1(\vec{r},t) = \vec{E}_01 \cos\left(\vec{k}_1 \cdot \vec{r} - \omega t + \phi_1\right) = \vec{E}_01 \cos\left(ks_1 - \omega t + \phi_1\right)
\]

\[
\vec{E}_2(\vec{r},t) = \vec{E}_02 \cos\left(\vec{k}_2 \cdot \vec{r} - \omega t + \phi_2\right) = \vec{E}_02 \cos\left(ks_2 - \omega t + \phi_2\right)
\]
Two-Beam Interference

At a given position $\vec{r}$ the wave $\vec{E}_p$ is the superposition of $\vec{E}_1$ and $\vec{E}_2$:

$$\vec{E}_p(\vec{r},t) = \vec{E}_1(\vec{r},t) + \vec{E}_2(\vec{r},t)$$

The irradiance $\left(W/m^2\right)$ is given by:

$$I = \varepsilon_0 c \langle \vec{E}_p^2 \rangle = \varepsilon_0 c \left[ \langle (\vec{E}_1 + \vec{E}_2) \cdot (\vec{E}_1 + \vec{E}_2) \rangle \right]$$

$$= \varepsilon_0 c \langle \vec{E}_1^2 + \vec{E}_2^2 + 2\vec{E}_1 \cdot \vec{E}_2 \rangle$$

$$I = I_1 + I_2 + I_{12}$$
Two-Beam Interference

The irradiances for beams 1 and 2 are given by:

\[ I_1 = \varepsilon_0 c \langle \bar{E}_1^2 \rangle \quad I_2 = \varepsilon_0 c \langle \bar{E}_2^2 \rangle \]

The interference term is given by:

\[ I_{12} = 2 \varepsilon_0 c \langle \bar{E}_1 \cdot \bar{E}_2 \rangle \]

\[ \bar{E}_1 \cdot \bar{E}_2 = \bar{E}_{01} \cdot \bar{E}_{02} \cos(ks_1 - \omega t + \phi_1) \cos(ks_2 - \omega t + \phi_2) \]

\[ 2 \langle \bar{E}_1 \cdot \bar{E}_2 \rangle = 2 \bar{E}_{01} \cdot \bar{E}_{02} \langle \cos(\alpha - \omega t) \cos(\beta - \omega t) \rangle \]

\[ = \bar{E}_{01} \cdot \bar{E}_{02} \left[ \langle \cos(\alpha + \beta - 2\omega t) \rangle + \langle \cos(\beta - \alpha) \rangle \right] \]

\[ = \bar{E}_{01} \cdot \bar{E}_{02} \langle \cos(\beta - \alpha) \rangle \]

\[ = \bar{E}_{01} \cdot \bar{E}_{02} \langle \cos \delta \rangle \quad \delta \equiv k \left( s_1 - s_2 \right) + (\phi_2 - \phi_1) : \text{phase difference} \]

\[ I = I_1 + I_2 + 2\sqrt{I_1 I_2} \langle \cos \delta \rangle \quad \text{when } \bar{E}_{01} \perp \bar{E}_{02} \text{ (same polarization)} \]

\[ I = I_1 + I_2 + 2\sqrt{I_1 I_2} \cos \delta \quad \text{for purely monochromatic light} \]
Interference of mutually incoherent beams

\[ I = I_1 + I_2 + 2\sqrt{I_1 I_2} \langle \cos \delta \rangle \]

\[ \langle \cos \delta \rangle = \left\langle \cos \left\{ k (s_1 - s_2) + \phi_2(t) - \phi_1(t) \right\} \right\rangle \]

“mutual incoherence” means \( \phi_1(t) \) and \( \phi_2(t) \) are random in time

\[ \langle \cos \delta \rangle = 0 \]

\[ I = I_1 + I_2 \]

Mutually incoherent beams do not interfere with each other
Interference of mutually coherent beams

\[ I = I_1 + I_2 + 2\sqrt{I_1 I_2} \langle \cos \delta \rangle \]

\[ \langle \cos \delta \rangle = \langle \cos \{ k(s_1 - s_2) + \phi_2(t) - \phi_1(t) \} \rangle \]

“mutual coherence” means \( [\phi_1(t) - \phi_2(t)] \) is constant in time

\[ \langle \cos \delta \rangle = \cos \delta = \cos \left\{ k(s_1 - s_2) \right\} \]

\[ I = I_1 + I_2 + 2\sqrt{I_1 I_2} \cos \delta \]

Mutually coherent beams do interference with each other
Interference of mutually coherent beams

The total irradiance is given by

\[ I = I_1 + I_2 + 2 \sqrt{I_1 I_2} \cos \delta \]

There is a maximum in the interference pattern when \( \delta = 0, \pm 2\pi, \ldots \)

\[ I_{\text{max}} = I_1 + I_2 + 2 \sqrt{I_1 I_2} \cos(2n\pi) = I_1 + I_2 + 2 \sqrt{I_1 I_2} \]

This is referred to as **constructive interference**.

There is a minimum in the interference pattern when \( \delta = \pm \pi, \pm 3\pi, \ldots \)

\[ I_{\text{min}} = I_1 + I_2 + 2 \sqrt{I_1 I_2} \cos[(2n+1)\pi] = I_1 + I_2 - 2 \sqrt{I_1 I_2} \]

This is referred to as **destructive interference**.
Visibility

Visibility = fringe contrast

\[ V \equiv \frac{I_{\text{max}} - I_{\text{min}}}{I_{\text{max}} + I_{\text{min}}} \quad \{ 0 \leq V \leq 1 \} \]

When \[ I_1 = I_2 = I_0 \quad \Rightarrow \quad I = 2I_0 + 2I_0 \cos \delta = 4I_0 \cos^2 \left( \frac{\delta}{2} \right) \]

\[ \Rightarrow \quad I_{\text{max}} = 4I_0 \quad I_{\text{min}} = 0 \]

Therefore, \( V = 1 \)
Conditions for good visibility

Sources must be:

- **same in phase evolution** in terms of:
  - time (source frequency) → temporal coherence
  - space (source size) → spatial coherence

- Same in amplitude

- Same in polarization

*Figure 9.4* Interference of polarized light.
Young’s Double Slit Experiment

Assume that $y \ll s$ and $a \ll s$.

The condition for an interference maximum is

$$S_2P - S_1P = \Delta = m\lambda \approx a\sin\theta$$

The condition for an interference minimum is

$$S_2P - S_1P = \Delta = \left(m + \frac{1}{2}\right)\lambda \approx a\sin\theta$$

Relation between geometric path difference and phase difference:

$$\delta = k\Delta = \left(\frac{2\pi}{\lambda}\right)\Delta$$
On the screen the irradiance pattern is given by

\[ I = 4I_0 \cos^2 \left( \frac{\delta}{2} \right) = 4I_0 \cos^2 \left( \frac{\pi \Delta}{\lambda} \right) = 4I_0 \cos^2 \left( \frac{\pi a \sin \theta}{\lambda} \right) \]

Assuming that \( y \ll s \):

\[ \sin \theta = \tan \theta = \frac{y}{s} \quad \Rightarrow \quad I = 4I_0 \cos^2 \left( \frac{\pi a y}{s \lambda} \right) \]

**Bright fringes:**

\[ y_{\text{max}} = \frac{m \lambda s}{a} \quad m = 0, \pm 1, \pm 2, \ldots \]

**Dark fringes:**

\[ y_{\text{min}} = \frac{(m + \frac{1}{2}) \lambda s}{a} \quad m = 0, \pm 1, \pm 2, \ldots \]

\[ \Delta y_{\text{max}} = \Delta y_{\text{min}} = \left( \frac{\lambda s}{a} \right) \]
Figure 7-5  Alternating bright and dark interference fringes are produced by light from two coherent sources. Along directions where crests (solid circles) from $S_1$ intersect crests from $S_2$, brightness (B) results. Along directions where crests meet valleys (dashed circles), darkness (D) results.
Interference Fringes From 2 Point Sources

Two coherent point sources: \( P_1 \) and \( P_2 \)

\[
d = \frac{|r_1 - r_2|}{\lambda}
\]

\[
v t_1 = s'_1
\]

\[
v t_2 = s'_2
\]
Interference With Virtual Sources: Fresnel’s Double Mirror

Interference With Virtual Sources: Lloyd’s Mirror

Figure 7-7  Lloyd’s mirror.
Interference With Virtual Sources: Fresnel’s Biprism

7-4. Interference in Dielectric Films

Figure 10-9  Double-beam interference from a film. Rays reflected from the top and bottom plane surfaces of the film are brought together at $P$ by a lens.
Analysis of Interference in Dielectric Films

Figure 7-12  Single-film interference with light incident at arbitrary angle $\theta_i$. 
The phase difference due to optical path length differences for the front and back reflections is given by

\[ \Delta_p = n_f \left( \frac{AB + BC}{AD} \right) - n_0 \left( \frac{AD}{AD} \right) \]

\[ = n_f \left( \frac{AE + FC}{AD} \right) + n_f \left( \frac{EB + BF}{AD} \right) \]

\[ n_0 \sin \theta_i = n_f \sin \theta_t \]

\[ AD = 2AG \sin \theta_i = 2AG \frac{n_f}{n_0} \sin \theta_t \]

\[ = \frac{n_f}{n_0} (A \chi + n \chi) \]

\[ \Rightarrow \Delta_p = + n_f \left( \frac{EB + BF}{AD} \right) = 2n_f \left( \frac{EB}{AD} \right) \]

\[ \Delta_p = 2n_f t \cos \theta_t \]
Analysis of Interference in Dielectric Films

Also need to account for phase differences $\Delta_r$ due to differences in the reflection process at the front and back surfaces $\Rightarrow \Delta_r$

$$\Delta = \Delta_p + \Delta_r$$

$$\Delta = 2n_f t \cos \theta_i + \Delta_r$$

**Constructive interference**

$$\Delta = \Delta_p + \Delta_r = m\lambda$$

$m = 0, \pm 1, \pm 2, \ldots$

**Destructive interference**

$$\Delta = \Delta_p + \Delta_r = \left( m + \frac{1}{2} \right) \lambda$$

$m = 0, \pm 1, \pm 2, \ldots$
Fringes of Equal Inclination

Fringes arise as $\Delta$ varies due to changes in the incident angle: $\theta_i \rightarrow \theta_t$.

$$\Delta = 2n_f t \cos \theta_i + \Delta_r$$

**Constructive interference**

$$\Delta = \Delta_p + \Delta_r = m\lambda$$

$m = 0, \pm 1, \pm 2, \ldots$

**Destructive interference**

$$\Delta = \Delta_p + \Delta_r = \left(m + \frac{1}{2}\right)\lambda$$

$m = 0, \pm 1, \pm 2, \ldots$

Figure 7-13: Interference by a dielectric film with an extended source. Fringes of equal inclination are focused by a lens.
7-5. Fringes of Equal Thickness

When the direction of the incoming light is fixed, fringes arise as $\Delta$ varies due to changes in the dielectric film thickness:

$\Delta = 2n_f t \cos \theta_t + \Delta_r$

**Constructive interference → Bright fringe**

$\Delta = \Delta_p + \Delta_r = m\lambda$

$m = 0, \pm 1, \pm 2,\ldots$

**Destructive interference → Dark fringe**

$\Delta = \Delta_p + \Delta_r = \left(m + \frac{1}{2}\right)\lambda$

$m = 0, \pm 1, \pm 2,\ldots$

*Figure 10-14* Interference from a wedge-shaped film, producing localized fringes of equal thickness. (a) Viewing assembly. (b) Air wedge formed with two microscope slides.
7-6. Fringes of Equal Thickness: Newton’s Rings

Figure 7-17  (a) Newton’s rings apparatus. Interference fringes of equal thickness are produced by the air wedge between lens and optical flat. (b) Essential geometry for production of Newton’s rings.
Newton’s rings

\[ R^2 = r_m^2 + (R - t_m)^2 \rightarrow R = \frac{r_m^2 + t_m^2}{2t_m} \]

\[ r_m^2 \approx 2R t_m \]

\[ \Delta = 2 n_f t \cos \theta_t + \Delta_r \]

Maxima (bright rings) when,
\[ \Delta = 2t_m + \frac{\lambda}{2} = m\lambda \quad (n_f = 1, \Delta_r = \frac{\lambda}{2}) \]

\[ \rightarrow r_m^2 = \left(m - \frac{1}{2}\right)\lambda R \]

Minima (dark rings) when,
\[ \Delta = 2t_m + \frac{\lambda}{2} = \left(m + \frac{1}{2}\right)\lambda \]

\[ \rightarrow r_m^2 = m\lambda R \]
7-7. Film-thickness measurement by interference

\[ \Delta_p + \Delta_r = 2nt + \Delta_r = m\lambda \]

now changes by an amount \( \Delta t = d \),

\[ 2n \Delta t = 2d = (\Delta m)\lambda \]

\[ d = (\Delta x/x)(\lambda/2) \]
7-8. Stokes Relations in reflection and transmission

$E_i$ is the amplitude of the incident light. The amplitudes of the reflected and transmitted beams are given by

\[ E_r = r E_i \quad \text{and} \quad E_t = t E_i \]

From the principle of reversibility

\[ E_i = (r^2 + t^{'2}) E_i \]
\[ 0 = (r^{'2} + t r) E_i \]

Stokes relations

\[ t^{'2} = 1 - r^2 \]
\[ r = -r^{'2} \quad \rightarrow \quad r = e^{i\pi} r^{'2} \]
7-9. Multiple-Beam Interference in a Parallel Plate

Transmittance:
\[ T \equiv \frac{I_T}{I_0} \]

Reflectance:
\[ R \equiv \frac{I_R}{I_0} \]
Find the superposition of the reflected beams from the top of the plate. The phase difference between neighboring beams is

$$\delta = k\Delta$$

$$\Delta = 2n_f d \cos \theta_t$$

Given that the incident wave is

$$E_0 \exp(i \omega t)$$

$$E_1 = (rE_0) \exp(i \omega t)$$

$$E_2 = (tt' r' E_0) \exp[i(\omega t - \delta)]$$

$$E_3 = [tt'(r')^3 E_0] \exp[i(\omega t - 2\delta)]$$

$$E_4 = [tt'(r')^5 E_0] \exp[i(\omega t - 3\delta)]$$

$$\vdots$$

$$E_N = [tt'(r')^{2N-3} E_0] \exp\{i[\omega t - (N-1)\delta]\}$$
The reflected amplitude resulting from the superposition of the reflected beams from the top of the plate is given by

\[
E_R = \sum_{N=1}^{\infty} E_N = (rE_0) \exp(i\omega t) + \sum_{N=2}^{\infty} \left[ t t' (r')^{2N-3} E_0 \right] \exp\left\{ i\omega t - (N-1)\delta \right\}
\]

\[
= E_0 \exp(i\omega t) \left\{ r + t t' r' \exp(-i\delta) \sum_{N=2}^{\infty} (r')^{2N-4} \exp\left[ -i(N-2)\delta \right] \right\}
\]

Define \( x = (r')^2 \exp(-i\delta) \)

\[
\sum_{N=2}^{\infty} (r')^{2(N-2)} \exp\left[ -i(N-2)\delta \right] = \sum_{N=2}^{\infty} x^{N-2} = 1 + x + x^2 + \cdots = \frac{1}{1-x}
\]

Therefore

\[
E_R = E_0 \exp(i\omega t) \left\{ r + \frac{t t' r' \exp(-i\delta)}{1 - (r')^2 \exp(-i\delta)} \right\}
\]
The *Stokes relations* can now be used to simplify the expression

\[
E_R = E_0 \exp(i \omega t) \left\{ r + \frac{tt' \exp(-i \delta)}{1 - (r')^2 \exp(-i \delta)} \right\}
\]

\[
= E_0 \exp(i \omega t) \left\{ r + \frac{(1-r^2)(-r)\exp(-i \delta)}{1 - r^2 \exp(-i \delta)} \right\}
\]

\[
= E_0 \exp(i \omega t) \left\{ \frac{r[1 - r^2 \exp(-i \delta)] - r(1-r^2)\exp(-i \delta)}{1 - r^2 \exp(-i \delta)} \right\}
\]

\[
= E_0 \exp(i \omega t) \left\{ \frac{r - r^3 \exp(-i \delta) - r \exp(-i \delta) + r^3 \exp(-i \delta)}{1 - r^2 \exp(-i \delta)} \right\}
\]

\[
E_R = E_0 \exp(i \omega t) \left\{ \frac{r[1 - \exp(-i \delta)]}{1 - (r)^2 \exp(-i \delta)} \right\}
\]
The reflection irradiance is given by

\[ I_R \propto E_R^* E_R = E_0^2 \exp(i \omega t) \exp(-i \omega t) \frac{r[1 - \exp(-i \delta)]}{1 - r^2 \exp(-i \delta)} \left\{ \frac{r[1 - \exp(+i \delta)]}{1 - r^2 \exp(+i \delta)} \right\} \]

\[ = E_0^2 \left\{ \frac{r^2[1 - \exp(-i \delta) - \exp(+i \delta) + 1]}{1 - r^2 \exp(-i \delta) - r^2 \exp(+i \delta) + (r)^4} \right\} \]

\[ \exp(-i \delta) + \exp(+i \delta) = 2 \cos \delta \]

\[ I_R = E_R^* E_R = E_0^2 \left\{ \frac{2r^2(1 - \cos \delta)}{1 - 2r^2 \cos \delta + r^4} \right\} \]

\[ \frac{I_R}{I_0} = \frac{2r^2(1 - \cos \delta)}{1 - 2r^2 \cos \delta + r^4} \]
Multiple-Beam Interference in a Parallel Plate

The transmitted irradiance is given by

\[
\frac{I_T}{I_0} = 1 - \frac{I_R}{I_0}
\]

\[
= 1 - \frac{2r^2 (1 - \cos \delta)}{1 - 2r^2 \cos \delta + r^4} = \frac{1 - 2r^2 \cos \delta + r^4 - 2r^2 + 2r^2 \cos \delta}{1 - 2r^2 \cos \delta + r^4}
\]

\[
= \frac{1 + r^4 - 2r^2}{1 - 2r^2 \cos \delta + r^4}
\]

\[
\frac{I_T}{I_0} = \frac{(1 - r^2)^2}{1 - 2r^2 \cos \delta + r^4}
\]
Multiple-Beam Interference in a Parallel Plate

\[ \frac{I_R}{I_0} = \frac{2r^2 (1 - \cos \delta)}{1 - 2r^2 \cos \delta + r^4} \quad \frac{I_T}{I_0} = \frac{(1 - r^2)^2}{1 - 2r^2 \cos \delta + r^4} \]

**Minima in reflected irradiance and maxima in transmitted irradiance occur when**

\[ \cos \delta = 1, \quad \delta = 2\pi m \quad \Rightarrow \Delta = 2n_f d \cos \theta_t = m\lambda \quad \Rightarrow \left( \frac{I_R}{I_0} \right)_{\text{min}} = 0 \]

**Minima in transmitted irradiance and maxima in reflected irradiance occur when**

\[ \cos \delta = -1, \quad \delta = 2\pi \left( m + \frac{1}{2} \right) \quad \Rightarrow \Delta = 2n_f d \cos \theta_t = \left( m + \frac{1}{2} \right)\lambda \]

\[ \left( \frac{I_T}{I_0} \right)_{\text{min}} = \frac{1 + r^4 - 2r^2}{1 - 2r^2 (-1) + r^4} = \frac{(1 - r^2)^2}{(1 + r^2)^2} \]

\[ \left( \frac{I_R}{I_0} \right)_{\text{max}} = \frac{2r^2 (1+1)}{1 - 2r^2 (-1) + r^4} = \frac{4r^2}{(1 + r^2)^2} \]