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Electromagnetic energy density in a dispersive and absorptive material

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Abstract

The energy density associated with an electromagnetic wave passing through a medium, in which both the permittivity and the permeability are dispersive and absorptive, is derived. The energy density formula is applied to the calculation of the energy transport velocity in a left-handed material.

Electromagnetic energy density in metals

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In a medium with no dispersion or losses (*ε and μ are real and frequency-independent),* the time averaged electromagnetic energy density is given by (assuming harmonic time dependence)

$$\langle W \rangle_t = \left\langle \frac{1}{2}\left(E\cdot D + B\cdot H\right)\right\rangle_t \longrightarrow \bar{W} = \frac{1}{4}\left(\varepsilon\varepsilon_0|E|^2 + \mu\mu_0|H|^2\right)$$

When the medium is dispersive, *$\varepsilon=\varepsilon(\omega)$ and $\mu=\mu(\omega)$,* such that the imaginary parts *are* ***not very small*** *in comparison with* their real parts, the average energy density ("***effective EM energy density***") reduces to

$$\bar{W} = \frac{\varepsilon_0}{4}\left(\varepsilon' + \frac{2\omega\varepsilon''}{\Gamma_e}\right)|E|^2 + \frac{\mu_0}{4}\left(\mu' + \frac{2\omega\mu''}{\Gamma_h}\right)|H|^2$$

Total energy densities stored in Electric and Magnetic fields for dispersive media

$$(\mu' = 1,\ \mu'' = 0) \Longrightarrow \bar{W} = \frac{\varepsilon_0}{4}\left(\varepsilon' + \frac{2\omega\varepsilon''}{\Gamma_e}\right)|E|^2 + \frac{\mu_0}{4}|H|^2$$

$$H = [\varepsilon(\omega)\varepsilon_0/\mu_0]^{1/2}E \Longrightarrow \bar{W} = \frac{\varepsilon_0}{2}\left(n^2 + \frac{2\omega n\kappa}{\Gamma_e}\right)|E|^2$$

$$\varepsilon(\omega) = (n + i\kappa)^2$$

Let's drive this effective energy density

When the medium is dispersive, *ε=ε(ω) and μ=μ(ω), but ε and μ are assumed as purely real,*

such that the imaginary parts of *ε(ω) and μ(ω) are very small in comparison with* their real parts, the average energy density is

$$\bar{W} = \langle W \rangle_t = \left\langle \frac{1}{2}(E \cdot D + B \cdot H) \right\rangle_t = \frac{1}{2}\mathrm{Re}\left[\frac{d\{\omega\varepsilon(\omega)\}}{d\omega}\right]\langle E \cdot E \rangle + \frac{1}{2}\mathrm{Re}\left[\frac{d\{\omega\mu(\omega)\}}{d\omega}\right]\langle H \cdot H \rangle$$

For a field consisting of monochromatic components assuming harmonic time dependence at ω_0,

$$\bar{W} = \frac{1}{4}\left[\left.\frac{d\{\omega\varepsilon(\omega)\}}{d\omega}\right|_{\omega=\omega_0} |E|^2 + \left.\frac{d\{\omega\mu(\omega)\}}{d\omega}\right|_{\omega=\omega_0} |H|^2\right]$$

When the imaginary parts of *ε=ε(ω) and μ=μ(ω),* become large, we need another approach.

From the Lorentz model of the electric and magnetic polarizations under an oscillating EM fields, the equations of motion of the two polarizations are,

$$\ddot{\vec{P}} + \Gamma_e \dot{\vec{P}} + \omega_r^2 \vec{P} = \varepsilon_0 \omega_p^2 \vec{E} \qquad \ddot{\vec{M}} + \Gamma_h \dot{\vec{M}} + \omega_0^2 \vec{M} = F\omega_0^2 \vec{H}$$

ω_r (ω_0) *: the resonance frequency of the electric (magnetic)* dipole oscillators,
Γ_e (Γ_h) *: the damping frequency*
ω_p (F) *: a measure of the interaction strength between* the oscillators and the electric (magnetic) field.

the electric susceptibility χ_e, defined by $\vec{P} = \chi_e \varepsilon_0 \vec{E}$

The relative permittivity (relative dielectric constant) and the relative magnetic permeability are given by

$$\varepsilon(\omega) = 1 + \chi_e = 1 + \frac{\omega_p^2}{\omega_r^2 - \omega^2 - i\Gamma_e\omega} \qquad \mu(\omega) = 1 + \chi_m = 1 - \frac{F\omega_0^2}{\omega^2 - \omega_0^2 + i\Gamma_h\omega}$$

Poynting's theorem: $\dfrac{\partial W}{\partial t} = -\nabla\cdot\vec{S} - \vec{J}\cdot\vec{E}$ $W = \frac{1}{2}\left(\vec{E}\cdot\vec{D} + \vec{H}\cdot\vec{B}\right), \quad \vec{S} = \vec{E}\times\vec{H}$

➔ ***C**onservation of energy for the electromagnetic field*

➔ **Relation of the time derivative of the energy density, *W* to the energy flow and the rate at which the fields do work.**

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \Rightarrow \vec{B}\cdot(\vec{\nabla}\times\vec{E}) = -\vec{B}\cdot\frac{\partial\vec{B}}{\partial t}$$

$$\vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t} \Rightarrow \vec{E}\cdot(\vec{\nabla}\times\vec{H}) = \vec{J}\cdot\vec{E} + \vec{E}\cdot\frac{\partial\vec{D}}{\partial t}$$

$$\vec{E}\cdot(\vec{\nabla}\times\vec{H}) - \vec{B}\cdot(\vec{\nabla}\times\vec{E}) = \vec{J}\cdot\vec{E} + \vec{E}\cdot\frac{\partial\vec{D}}{\partial t} + \vec{B}\cdot\frac{\partial\vec{B}}{\partial t}$$

$$-\vec{\nabla}\cdot(\vec{E}\times\vec{H}) = \vec{J}\cdot\vec{E} + \vec{E}\cdot\frac{\partial\vec{D}}{\partial t} + \vec{B}\cdot\frac{\partial\vec{B}}{\partial t}$$

$$\longrightarrow \quad \vec{S} = \vec{E}\times\vec{H} \quad \longrightarrow \quad \vec{\nabla}\cdot\vec{S} + \vec{E}\cdot\frac{\partial\vec{D}}{\partial t} + \vec{B}\cdot\frac{\partial\vec{B}}{\partial t} = -\vec{J}\cdot\vec{E} \quad \Rightarrow \quad \frac{\partial W}{\partial t} + \vec{\nabla}\cdot\vec{S} + \vec{J}\cdot\vec{E} = 0$$

For $\vec{J} = 0$,

$$\vec{D} = \varepsilon_0\vec{E} + \vec{P}, \quad \vec{B} = \mu_0\vec{H} + \vec{M}$$

$$\int_\sigma (\vec{E}\times\vec{H})\cdot d\vec{\sigma} = -\int_V \left[\vec{E}\cdot\frac{\partial\vec{D}}{\partial t} + \vec{B}\cdot\frac{\partial\vec{B}}{\partial t}\right] dV = -\int_V \left[\varepsilon_0\left(\vec{E}\cdot\dot{\vec{E}}\right) + \left(\vec{E}\cdot\dot{\vec{P}}\right) + \mu_0\left(\vec{H}\cdot\dot{\vec{H}}\right) + \left(\vec{H}\cdot\dot{\vec{M}}\right)\right] dV$$

From Maxwell's equations,
the surface integral of the Poynting vector can be expressed as, (➔ ***Poynting's theorem***)

$$\int_\sigma (\vec{E}\times\vec{H})\cdot d\vec{\sigma} = -\int_V \left[\varepsilon_0(\vec{E}\cdot\dot{\vec{E}}) + (\vec{E}\cdot\dot{\vec{P}}) + \mu_0(\vec{H}\cdot\dot{\vec{H}}) + \mu_0(\vec{H}\cdot\dot{\vec{M}})\right] dV$$

$$(\vec{E}\cdot\dot{\vec{P}}) = \frac{1}{\varepsilon_0\omega_p^2}(\ddot{\vec{P}} + \Gamma_e\dot{\vec{P}} + \omega_r^2\vec{P})\cdot\dot{\vec{P}} = \frac{1}{2\varepsilon_0\omega_p^2}\frac{d}{dt}(\dot{P}^2 + \omega_r^2 P^2) + \frac{\Gamma_e}{\varepsilon_0\omega_p^2}\dot{P}^2$$

$$(\vec{H}\cdot\dot{\vec{M}}) = \frac{1}{F\omega_0^2}(\ddot{\vec{M}} + \Gamma_h\dot{\vec{M}} + \omega_0^2\vec{M})\cdot\dot{\vec{M}} = \frac{1}{2F\omega_0^2}\frac{d}{dt}(\dot{M}^2 + \omega_0^2 M^2) + \frac{\Gamma_h}{F\omega_0^2}\dot{M}^2$$

$$\int_\sigma (\vec{E}\times\vec{H})\cdot d\vec{\sigma} + \int_V \left(\frac{\Gamma_e\dot{P}^2}{\varepsilon_0\omega_p^2} + \frac{\mu_0\Gamma_h\dot{M}^2}{F\omega_0^2}\right) dV = -\int_V \dot{W}\, dV$$

Rate of energy loss *by leakage across* its surface

Rate of energy loss by Dissipation in the volume

Rate of change of the energy stored within the volume.

where, the energy density *W is defined by*

$$W = \frac{\varepsilon_0}{2}E^2 + \frac{\mu_0}{2}H^2 + \frac{1}{2\varepsilon_0\omega_p^2}(\dot{P}^2 + \omega_r^2 P^2) + \frac{\mu_0}{2F\omega_0^2}(\dot{M}^2 + \omega_0^2 M^2)$$

$$W = \frac{\varepsilon_0}{2}E^2 + \frac{\mu_0}{2}H^2 + \frac{1}{2\varepsilon_0\omega_p^2}\left(\dot{P}^2 + \omega_r^2 P^2\right) + \frac{\mu_0}{2F\omega_0^2}\left(\dot{M}^2 + \omega_0^2 M^2\right)$$

For harmonic time dependence the time average of the average energy density is

$$\bar{W} = \frac{\varepsilon_0}{4}|E|^2 + \frac{\mu_0}{4}|H|^2 + \frac{1}{4\varepsilon_0\omega_p^2}(\omega^2 + \omega_r^2)|P|^2 + \frac{\mu_0}{4F\omega_0^2}(\omega^2 + \omega_0^2)|M|^2$$

Expressing the polarization and the magnetization in terms of the electric and magnetic fields,

$$\bar{W} = \frac{\varepsilon_0}{4}\left[1 + \frac{(\omega^2 + \omega_r^2)\omega_p^2}{(\omega_r^2 - \omega^2)^2 + \Gamma_e^2\omega^2}\right]|E|^2 + \frac{\mu_0}{4}\left[1 + \frac{(\omega^2 + \omega_0^2)F\omega_0^2}{(\omega^2 - \omega_0^2)^2 + \Gamma_h^2\omega^2}\right]|H|^2$$

$$= \frac{\varepsilon_0}{4}\left(\varepsilon' + \frac{2\omega\varepsilon''}{\Gamma_e}\right)|E|^2 + \frac{\mu_0}{4}\left(\mu' + \frac{2\omega\mu''}{\Gamma_h}\right)|H|^2$$

$$\varepsilon(\omega) = 1 + \chi_e = 1 + \frac{\omega_p^2}{\omega_r^2 - \omega^2 - i\Gamma_e\omega}$$

$$\mu(\omega) = 1 + \chi_m = 1 - \frac{F\omega_0^2}{\omega^2 - \omega_0^2 + i\Gamma_h\omega}$$

In the case of no magnetic dispersion (*μ' = 1, μ" = 0)*

$$\bar{W} = \frac{\varepsilon_0}{4}\left(\varepsilon' + \frac{2\omega\varepsilon''}{\Gamma_e}\right)|E|^2 + \frac{\mu_0}{4}|H|^2$$

We have arrived!

Using $H = [\varepsilon(\omega)\varepsilon_0/\mu_0]^{1/2}E$ and $\varepsilon(\omega) = (n + i\kappa)^2$,

$$\bar{W} = \frac{\varepsilon_0}{2}\left(n^2 + \frac{2\omega n\kappa}{\Gamma_e}\right)|E|^2$$

As an example of the application of the general energy density expression of,

$$\bar{W} = \frac{\varepsilon_0}{4}\left(\varepsilon' + \frac{2\omega\varepsilon''}{\Gamma_e}\right)|E|^2 + \frac{\mu_0}{4}\left(\mu' + \frac{2\omega\mu''}{\Gamma_h}\right)|H|^2$$

Let's evaluate the **velocity of energy transport** in a composite material which is left-handed over a band of frequencies.

From the dispersion relation of EM waves, the group velocity *in a non-absorbing medium* is

$$k^2 = \varepsilon(\omega)\mu(\omega)\frac{\omega^2}{c^2} \longrightarrow v_G = d\omega/dk = \frac{2c\sqrt{\varepsilon(\omega)\mu(\omega)}}{2\varepsilon(\omega)\mu(\omega) + \omega\mu(\omega)\frac{d\varepsilon}{d\omega} + \omega\varepsilon(\omega)\frac{d\mu}{d\omega}}$$

On the other hand, the exact definition of the velocity *in any medium* is

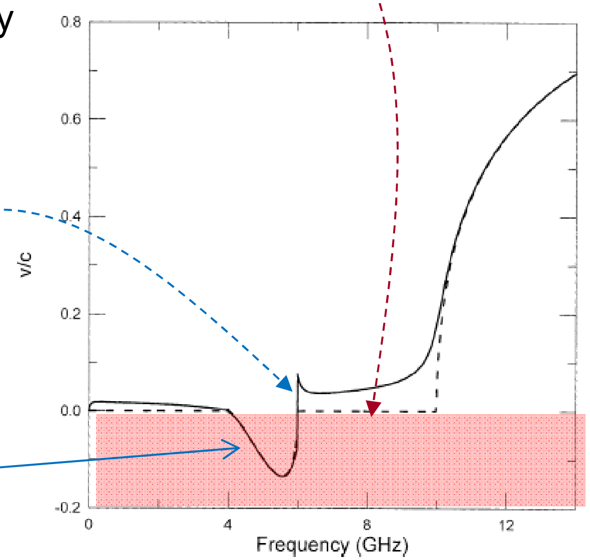
$$v_E = (-1)^p \frac{\bar{S}}{\bar{W}}$$ (*p* = *+1* for a right-handed medium, −1 for a left-handed one)

The average power flow is obtained from the complex Poynting vector by

$$\bar{S} = \frac{1}{2}\mathrm{Re}\langle\vec{E}\times\vec{H}^*\rangle \longrightarrow \bar{S} = \frac{1}{2}|E|^2\,\mathrm{Re}\left(\sqrt{\frac{\varepsilon\varepsilon_0}{\mu\mu_0}}\right)$$

For plane waves

$$v_E = (-1)^p \frac{2c\,\mathrm{Re}(\sqrt{\varepsilon/\mu}\,)}{(\varepsilon' + 2\omega\varepsilon''/\Gamma_e) + (\mu' + 2\omega\mu''/\Gamma_m)|\varepsilon/\mu|}$$



Negative group velocity

(Griffith) 8.1.2 Poynting's Theorem

In Chapter 2, we found that the **work necessary to assemble a static charge distribution** (against the Coulomb repulsion of like charges) is (Eq. 2.45)

Energy of Continuous Charge Distribution ⇨ $W = \frac{1}{2}\int \rho V \, d\tau$ ⇨ $W_e = \frac{\epsilon_0}{2}\int E^2 \, d\tau$

Likewise, the **work required to get currents going (against the back emf)** is (Eq. 7.34)

Energy of steady Current flowing ⇨ $W = \frac{1}{2} I \oint \mathbf{A} \cdot d\mathbf{l}$ ⇨ $W_m = \frac{1}{2\mu_0}\int B^2 \, d\tau$

Therefore, *the total energy stored in electromagnetic fields* is

⇨
$$U_{em} = \frac{1}{2}\int \left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\right) d\tau$$

➔ ***Let's derive this total energy stored in EM fields more generally in the context of the energy conservation law for electrodynamics.***

➔ ***"Energy conservation law for electrodynamics": Poynting Theorem***

Energy Conservation and Poynting's Theorem

Suppose we have some charge and current configuration which, at time *t,* produces fields **E** and **B**.
In the next instant, ***dt,*** the charges move around a bit.
→ ***How much work, dW, is done by the electromagnetic forces*** acting on these charges in the interval ***dt*** ?

According to the Lorentz force law, the work done on a charge *q* is

$$dW = \mathbf{F}\cdot d\mathbf{l} = q(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot\mathbf{v}\,dt = q\mathbf{E}\cdot\mathbf{v}\,dt$$

$$q = \rho d\tau \qquad \rho\mathbf{v} = \mathbf{J}$$

$$\frac{dW}{dt} = \int_{\mathcal{V}} (\mathbf{E}\cdot\mathbf{J})\,d\tau$$

$\mathbf{E}\cdot\mathbf{J}$ → **the work done per unit time, per unit volume, or, the *power* delivered per unit volume.**

Ampere-Maxwell law → $\mathbf{E}\cdot\mathbf{J} = \frac{1}{\mu_0}\mathbf{E}\cdot(\nabla\times\mathbf{B}) - \epsilon_0\mathbf{E}\cdot\frac{\partial\mathbf{E}}{\partial t}$

$\nabla\cdot(\mathbf{E}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{E}) - \mathbf{E}\cdot(\nabla\times\mathbf{B})$ and Faraday's law $(\nabla\times\mathbf{E} = -\partial\mathbf{B}/\partial t)$,

$$\mathbf{E}\cdot(\nabla\times\mathbf{B}) = -\mathbf{B}\cdot\frac{\partial\mathbf{B}}{\partial t} - \nabla\cdot(\mathbf{E}\times\mathbf{B})$$

$$\mathbf{B}\cdot\frac{\partial\mathbf{B}}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}(B^2), \quad \text{and} \quad \mathbf{E}\cdot\frac{\partial\mathbf{E}}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}(E^2)$$

$$\mathbf{E}\cdot\mathbf{J} = -\frac{1}{2}\frac{\partial}{\partial t}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) - \frac{1}{\mu_0}\nabla\cdot(\mathbf{E}\times\mathbf{B})$$

$$\frac{dW}{dt} = -\frac{d}{dt}\int_{\mathcal{V}}\frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right)d\tau - \frac{1}{\mu_0}\oint_{\mathcal{S}}(\mathbf{E}\times\mathbf{B})\cdot d\mathbf{a}$$

➔ **Poynting's theorem**

➔ **This is "Work-Energy Theorem" or "Energy Conservation Theorem" of Electrodynamics.**

Poynting's Theorem and Poynting Vector

$$\frac{dW}{dt} = -\frac{d}{dt}\int_{\mathcal{V}} \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) d\tau - \frac{1}{\mu_0}\oint_{\mathcal{S}} (\mathbf{E}\times\mathbf{B})\cdot d\mathbf{a}$$

➔ *Poynting's theorem*
➔ *Work-Energy Theorem or Energy Conservation Theorem of Electrodynamics.*

The first integral on the right is the total energy stored in the fields → $\int_{\mathcal{V}} \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) d\tau = U_{\text{em}}$

The second term evidently represents the rate at which energy is carried out of *V,* across its boundary surface, by the fields. → $\frac{1}{\mu_0}\oint_{\mathcal{S}} (\mathbf{E}\times\mathbf{B})\cdot d\mathbf{a}$

Poynting's theorem says

➔ ***"the work done on the charges by the electromagnetic force is equal to the decrease in energy stored in the field, less the energy that flowed out through the surface."***

The *energy per unit time, per unit area,* transported by the fields is called the *Poynting vector*:

$$\mathbf{S} \equiv \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B}) \quad (\mathrm{W/m^2})$$ ➔ ***Poynting vector***

$$\mathbf{S} \equiv (\mathbf{E}\times\mathbf{H})$$ ➔ ***Poynting vector in linear media***

Poynting's theorem ➔ $$\frac{dW}{dt} = -\frac{dU_{\text{em}}}{dt} - \oint_{\mathcal{S}} \mathbf{S}\cdot d\mathbf{a}$$

→ **S** · ***da*** is the energy per unit time crossing the infinitesimal surface ***da***
→ the energy *flux,* if you like (so ***S*** is the energy flux density).

Poynting's Theorem and Poynting Vector

$$\frac{dW}{dt} = -\frac{dU_{\text{em}}}{dt} - \oint_{\mathcal{S}} \mathbf{S} \cdot d\mathbf{a} \qquad \frac{dW}{dt} = \int_{\mathcal{V}} (\mathbf{E} \cdot \mathbf{J})\, d\tau \qquad U_{\text{em}} = \frac{1}{2}\int \left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\right) d\tau \qquad \mathbf{S} \equiv \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B})$$

The work ***W*** done on the charges by the fields will increase their mechanical energy (kinetic, potential, or whatever).

→ If we let u_{mech} denote the mechanical energy density,

$$\frac{dW}{dt} = \int_{\mathcal{V}} (\mathbf{E} \cdot \mathbf{J})\, d\tau = \frac{d}{dt} \int_{\mathcal{V}} u_{\text{mech}}\, d\tau$$

→ If we let u_{em} denote the electromagnetic energy density,

$$U_{em} = \int_V u_{em}\, d\tau \quad \rightarrow \quad u_{em} = \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{1}{\mu_0} B^2\right) = \frac{1}{2}\left(\vec{E} \cdot \vec{D} + \vec{H} \cdot \vec{B}\right)$$

$$\frac{dW}{dt} = -\frac{dU_{\text{em}}}{dt} - \oint_{\mathcal{S}} \mathbf{S} \cdot d\mathbf{a} \longrightarrow \frac{d}{dt} \int_{\mathcal{V}} (u_{\text{mech}} + u_{\text{em}})\, d\tau = -\oint_{\mathcal{S}} \mathbf{S} \cdot d\mathbf{a} = -\int_{\mathcal{V}} (\nabla \cdot \mathbf{S})\, d\tau$$

$$\boxed{\frac{\partial}{\partial t}(u_{\text{mech}} + u_{\text{em}}) = -\nabla \cdot \mathbf{S}}$$ ➔ ***differential version of Poynting's theorem***

➔ Compare it with the continuity equation, expressing conservation of charge: $\frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{J}$

➔ The charge density is replaced by the energy density (mechanical plus electromagnetic),
➔ the current density is replaced by the Poynting vector.

➔ ***Therefore, Poynting's theorem represents the flow of energy***
in in exactly the same way that **J** in the continuity equation describes the flow of charge.

Poynting's Theorem is the "Work-energy theorem" or "Conservation of Energy"

$$\frac{dW}{dt} = -\frac{d}{dt}\int_{\mathcal{V}} \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) d\tau - \frac{1}{\mu_0}\oint_{\mathcal{S}} (\mathbf{E}\times\mathbf{B})\cdot d\mathbf{a} \iff \frac{dW}{dt} = -\frac{dU_{\text{em}}}{dt} - \oint_{\mathcal{S}} \mathbf{S}\cdot d\mathbf{a} \iff \frac{\partial}{\partial t}(u_{\text{mech}} + u_{\text{em}}) = -\nabla\cdot\mathbf{S}$$

$$\boxed{\frac{dW}{dt} = -\frac{dU_{em}}{dt} - \oint_S S\cdot ds}$$

$S \equiv \frac{1}{\mu}(E\times B)$: Poynting vetor

Work done by the EM field

Total energy stored in the EM field

Energy flowed out through the surface

"The work done on the charges by the electromagnetic force is equal to the decrease in energy stored in the field, less the energy that flowed out through the surface" .

$$\frac{dW}{dt} = \int_V (E\cdot J)\,dv = -\frac{d}{dt}\int_V \left(\frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2\mu_0}B^2\right)dv - \frac{1}{\mu_0}\oint_S (E\times B)\cdot ds$$

$$\frac{\partial u_{em}}{\partial t} = -\nabla\cdot S - E\cdot J$$ ➔ ***differential version of Poynting's theorem***

Poynting's theorem $\frac{\partial u_{em}}{\partial t} = -\nabla \cdot \mathbf{S} - \mathbf{E} \cdot \mathbf{J}$ $\mathbf{S} = \mathbf{E} \times \mathbf{H}$

Let's prove it directly from Maxwell's equations

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \Rightarrow \vec{H} \cdot (\vec{\nabla} \times \vec{E}) = -\vec{H} \cdot \frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \Rightarrow \vec{E} \cdot (\vec{\nabla} \times \vec{H}) = \vec{E} \cdot \vec{J} + \vec{E} \cdot \frac{\partial \vec{D}}{\partial t}$$

$$\vec{E} \cdot (\vec{\nabla} \times \vec{H}) - \vec{H} \cdot (\vec{\nabla} \times \vec{E}) = \vec{E} \cdot \vec{J} + \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} + \vec{H} \cdot \frac{\partial \vec{B}}{\partial t}$$

$$-\vec{\nabla} \cdot (\vec{E} \times \vec{H}) = \vec{E} \cdot \vec{J} + \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} + \vec{H} \cdot \frac{\partial \vec{B}}{\partial t}$$

$$\vec{S} \equiv \vec{E} \times \vec{H} \longrightarrow \vec{\nabla} \cdot \vec{S} + \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} + \vec{H} \cdot \frac{\partial \vec{B}}{\partial t} = -\vec{E} \cdot \vec{J}$$

$$\Rightarrow \frac{\partial u_{em}}{\partial t} + \vec{\nabla} \cdot \vec{S} + \vec{E} \cdot \vec{J} = 0$$

: Poynting's theorem

For $\vec{J} = 0$ (in free space), $\Rightarrow \frac{\partial u_{em}}{\partial t} = -\vec{\nabla} \cdot \vec{S}$

For a steady state $\frac{\partial u_{em}}{\partial t} = 0$, $\Rightarrow -\vec{\nabla} \cdot \vec{S} = \vec{E} \cdot \vec{J}$