Chapter 12.
Electrodynamics and Relativity

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Does the principle of relativity apply to the laws of electrodynamics?
12.1 The Special Theory of Relativity

Does the principle of relativity apply to the laws of electrodynamics?

Take, for example, the reciprocal electrodynamic action of a magnet and a conductor:

The observable phenomenon here depends only on the relative motion of the conductor and the magnet.

If the magnet is stationary and the conductor in motion, no electric field arises in the neighborhood of the magnet. In the conductor, however, we find an magnetic force (motional emf).

On the other hand, for someone on the train the magnet is in motion and the conductor at rest. No magnetic force, because the conductor is at rest. But, a changing magnetic field induces an electric field by Faraday’s law, resulting in an electric force (induced emf).

- The motional emf and the induced emf give rise to electric currents of the same path and intensity.

- Was the equality of the two emf's Just a lucky accident?

Einstein could not believe this was a mere coincidence; he took it, rather, as a clue that electromagnetic phenomena, like mechanical ones, obey the principle of relativity.
Einstein proposed his two famous postulates:

1. **The principle of relativity.** (first elevated by Galileo Galilei)
   - The laws of physics apply in all inertial reference systems.
   - It states that there is no absolute rest system.

2. **The universal speed of light.**
   - The speed of light in vacuum is the same for all inertial observers, regardless of the motion of the source.

   **Einstein’s velocity addition rule:**
   \[ v_{AC} = \frac{v_{AB} + v_{BC}}{1 + (v_{AB}v_{BC}/c^2)} \]

   **Galileo’s velocity addition rule:**
   \[ v_{AC} = v_{AB} + v_{BC} \]

The three most striking geometrical consequences of Einstein’s postulates:

- **the relativity of simultaneity:** Two events that are simultaneous in one inertial system are not, in general, simultaneous in another.

- **time dilation:** Moving clocks run slow.
  \[ \Delta T = \frac{1}{\sqrt{1 - v^2/c^2}} \Delta t = \frac{1}{\gamma} \Delta t \]

  \[ \gamma \equiv \frac{1}{\sqrt{1 - v^2/c^2}} \]

  \( \Delta T \) (measured on car) \( \Delta t \) (measured on ground)

- **Lorentz contraction:** Moving objects looks shorter, *from ground point of view.

  \[ \Delta x = \frac{1}{\sqrt{1 - v^2/c^2}} \Delta x = \gamma \Delta x \]
12.1 The Special Theory of Relativity

(i) The relativity of simultaneity: Two events that are simultaneous in one inertial system are not, in general, simultaneous in another.

(ii) Time dilation: Moving clocks run slow.

\[ \Delta \tilde{T} = \sqrt{1 - \frac{v^2}{c^2}} \Delta t \]

The twin paradox.

(iii) Lorentz (length) contraction: Moving objects looks shorter.

Only along the direction of its motion.
Dimensions perpendicular to the velocity are not contracted.

\[ \Delta \tilde{x} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \Delta x \]
12.1.3 The Lorentz Transformations

\( d \) is the distance from \( \tilde{O} \) to \( \tilde{A} \) as measured in \( \tilde{S} \)
\( \tilde{x} \) is the distance from \( \tilde{O} \) to \( \tilde{A} \) as measured in \( \tilde{S} \)

**Galilean transformations**

\[
\begin{align*}
d &= \tilde{x},  \\
x &= d + vt
\end{align*}
\]

\[
\begin{align*}
(i) & \quad \tilde{x} = x - vt, \\
(ii) & \quad \tilde{y} = y, \\
(iii) & \quad \tilde{z} = z, \\
(iv) & \quad \tilde{t} = t.
\end{align*}
\]

**Lorentz transformations**

\[
\begin{align*}
d &= \frac{1}{\gamma} \tilde{x},  \\
x &= d + vt
\end{align*}
\]

from the point of view of \( \tilde{S} \)
\[
x = \gamma (\tilde{x} + vt), \text{ since } \tilde{x} = \gamma (x - vt)
\]
12.1.3 The Lorentz Transformations

(i) \( \tilde{x} = \gamma(x - vt) \), \( \Delta x = \frac{1}{\gamma} \Delta \tilde{x} \) \( \rightarrow \) Moving object looks shorter.

(ii) \( \tilde{y} = y \),

(iii) \( \tilde{z} = z \),

(iv) \( \tilde{t} = \gamma \left(t - \frac{v}{c^2}x\right) \) \( \rightarrow \Delta t = \frac{1}{\gamma} \Delta \tilde{t} \) \( \rightarrow \) Moving clocks run slow.

(Ex) Einstein's velocity addition rule

In \( S \), suppose a particle moves a distance \( dx \) in a time \( dt \):

\[ \text{Velocity} \rightarrow u = \frac{dx}{dt} \]

In \( \tilde{S} \), meanwhile, it has moved

\[ d\tilde{x} = \gamma(dx - vdt) \]
\[ d\tilde{t} = \gamma \left(dt - \frac{v}{c^2}dx\right) \]

The velocity in \( \tilde{S} \) is therefore

\[ \tilde{u} = \frac{d\tilde{x}}{d\tilde{t}} = \frac{\gamma(dx - vdt)}{\gamma \left(dt - \frac{v}{c^2}dx\right)} = \frac{(dx/dt - v)}{1 - \frac{v}{c^2}dx/dt} = \frac{u - v}{1 - uv/c^2} \]
12.1.4 The Structure of Spacetime

**Four-vectors** \((x^0, x^1, x^2, x^3)\)

\[ x^0 = ct \quad \text{Using } x^0 \text{ (instead of } t) \quad \text{to changing the unit of time from the second to the meter} \]

\[ x^1 = x \]

\[ x^2 = y \quad \beta \equiv \frac{v}{c} \quad \text{Using } \beta \text{ (instead of } v) \]

\[ x^3 = z \]

**Lorentz transformation matrix:** \(\Lambda\)

\[
\begin{align*}
\text{(i)} & \quad \bar{x} = \gamma(x - vt), \\
\text{(ii)} & \quad \bar{y} = y, \\
\text{(iii)} & \quad \bar{z} = z, \\
\text{(iv)} & \quad \bar{t} = \gamma \left( t - \frac{v}{c^2} x \right) \\
\end{align*}
\]

\[
\begin{pmatrix}
\bar{x}^0 \\
\bar{x}^1 \\
\bar{x}^2 \\
\bar{x}^3
\end{pmatrix} =
\begin{pmatrix}
\gamma & -\gamma \beta & 0 & 0 \\
-\gamma \beta & \gamma & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
x^0 \\
x^1 \\
x^2 \\
x^3
\end{pmatrix}
\]

\[
\bar{x}^\mu = \sum_{\nu=0}^{3} (\Lambda^\mu_\nu) x^\nu
\]
12.1.4 The Structure of Spacetime

**Four-vectors** = any set of four components that transform under Lorentz transformations

\[
\tilde{a}^\mu = \sum_{\nu=0}^{3} \Lambda^\mu_\nu a^\nu
\]

\[
\begin{align*}
\tilde{a}^0 &= \gamma (a^0 - \beta a^1), \\
\tilde{a}^1 &= \gamma (a^1 - \beta a^0), \\
\tilde{a}^2 &= a^2, \\
\tilde{a}^3 &= a^3.
\end{align*}
\]

**Four-dimensional scalar product:** \(A \cdot B \equiv -a^0 b^0 + a^1 b^1 + a^2 b^2 + a^3 b^3\)

(Prove; Prob. 12.17) The 4-dim. Dot product has the same value in all inertial systems:

\[-\tilde{a}^0 \tilde{b}^0 + \tilde{a}^1 \tilde{b}^1 + \tilde{a}^2 \tilde{b}^2 + \tilde{a}^3 \tilde{b}^3 = -a^0 b^0 + a^1 b^1 + a^2 b^2 + a^3 b^3\]

**Einstein summation convention**

To keep track of the minus sign it is convenient to introduce the **covariant** vector \(a_\mu\), which differs from the **contravariant** \(a^\mu\) only in the sign of the zeroth component:

\[
a_\mu = (a_0, a_1, a_2, a_3) = (-a^0, a^1, a^2, a^3)
\]

\[
A \cdot B \equiv -a^0 b^0 + a^1 b^1 + a^2 b^2 + a^3 b^3 = a_0 b^0 + a_1 b^1 + a_2 b^2 + a_3 b^3 = \sum_{\mu=0}^{3} a_\mu b^\mu
\]

\[
A \cdot B \equiv \sum_{\mu=0}^{3} a_\mu b^\mu \rightarrow a_\mu b^\mu
\]
The invariant interval in spacetime

Suppose event A occurs at \((x^0_A, x^1_A, x^2_A, x^3_A)\), and event B at \((x^0_B, x^1_B, x^2_B, x^3_B)\).

**Displacement 4-vector:**

\[
\Delta x^\mu = x^\mu_A - x^\mu_B
\]

**Interval between two events:** scalar product of \(\Delta x^\mu\) with itself

\[
I \equiv (\Delta x)_\mu (\Delta x)^\mu = - (\Delta x^0)^2 + (\Delta x^1)^2 + (\Delta x^2)^2 + (\Delta x^3)^2 = -c^2 t^2 + d^2
\]

where \(t\) is the time difference between the two events and \(d\) is their spatial separation.

When you transform to a moving system, the time between A and B, and the spatial separation are altered. **But, the interval I remains the same!**

Depending on the two events in question, the interval can be positive, negative, or zero:

- **timelike** 1. If \(I < 0\) we call the interval timelike, for this is the sign we get when the two occur at the same place \((d = 0)\), and are separated only temporally.

- **spacelike** 2. If \(I > 0\) we call the interval spacelike, for this is the sign we get when the two occur at the same time \((t = 0)\), and are separated only spatially.

- **lightlike** 3. If \(I = 0\) we call the interval lightlike, for this is the relation that holds when the two events are connected by a signal traveling at the speed of light.
Space-time diagrams (Minkowski diagrams) and World line

- Timelike (slope greater than 1)
- Spacelike (slope less than 1)
- Lightlike (slope 1)

Trajectory of a particle on a Minkowski diagram → world line.