2.1 The Electric Field

2.1.1 Introduction

The fundamental problem that electromagnetic theory hopes to solve is this:
→ **What force** do the **source charges** \((q_1, q_2, \ldots)\) exert on the **test charge** \((Q)\)?
→ In general, **both the source charges and the test charge are in motion**.

The solution to this problem is facilitated by the **principle of superposition**
→ The interaction between any two charges is **completely unaffected by the others**.
→ To determine the force on \(Q\), we can first compute the force \(F_1\), due to \(q_1\) alone (ignoring all the others); then we compute the force \(F_2\), due to \(q_2\) alone; and so on.
→ Finally, we take the vector sum of all these individual forces: \(F = F_1 + F_2 + F_3 + \ldots\)

To solve the force on \(Q\) using the superposition principle sounds very easy, BUT,
→ the force on \(Q\) depends **not only on the separation distance** \(r\) **between the charges**,  
→ it also depends on **both their velocities and on the acceleration** of \(q\).

→ **Moreover**, electromagnetic fields travels at the speed of light, so the position, velocity, and acceleration that \(q\) **had** at some earlier time, affect the force at present.

To begin with, **consider the special case of ELECTROSTATICS**
→ **All the source charges are STATIONARY**
→ **The test charge may be moving.**
2.1.2 Coulomb's Law

What is the force on a test charge $Q$ due to a single point charge $q$ which is at *rest* a distance $r$ away?

![Diagram of a charge $Q$ and a point charge $q$ with vectors $r$ and $r'$]

The experimental law of Coulomb (1785)

$$\mathbf{F} = \frac{1}{4\pi \epsilon_0} \frac{qQ}{r^2} \hat{r}$$

- $\epsilon_0 = 8.85 \times 10^{-12} \frac{C^2}{N \cdot m^2}$: permittivity of free space
2.1.3 The Electric Field

If we have *several* point charges \( q_1, q_2, \ldots, q_n \), at distances \( \mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n \) from \( Q \), the total force on \( Q \) is evidently

\[
F = F_1 + F_2 + \ldots = \frac{1}{4\pi \epsilon_0} \left( \frac{q_1 Q}{\mathbf{r}_1^2} \mathbf{\hat{r}}_1 + \frac{q_2 Q}{\mathbf{r}_2^2} \mathbf{\hat{r}}_2 + \ldots \right)
\]

\[
= \frac{Q}{4\pi \epsilon_0} \left( \frac{q_1}{\mathbf{r}_1^2} \mathbf{\hat{r}}_1 + \frac{q_2}{\mathbf{r}_2^2} \mathbf{\hat{r}}_2 + \frac{q_3}{\mathbf{r}_3^2} + \ldots \right)
\]

\[
F = QE
\]

\[
E(\mathbf{r}) \equiv \frac{1}{4\pi \epsilon_0} \sum_{i=1}^{n} \frac{q_i}{\mathbf{r}_i^2} \mathbf{\hat{r}}_i
\]

\( E \) is called the **Electric Field** of the source charges.

\( E(\mathbf{r}) \) is the **force per unit charge** that would be exerted on a test charge, if you were to place one at \( P \).

Notice that it is a function of position \( (\mathbf{r}) \), because the separation vectors \( \mathbf{r} \) depend on the location of the field point \( P \).

It makes no reference to the test charge \( Q \).
2.1.4 Continuous Charge Distributions

If the charge is distributed continuously over some region, the sun becomes an integral:

\[ E(r) = \frac{1}{4\pi \varepsilon_0} \int \frac{1}{r^2} \hat{r} \, dq \]

→ If the charge is spread out along a line with charge-per-unit-length \( \lambda \),

\[ dq = \lambda \, dl' \quad \rightarrow \quad E(r) = \frac{1}{4\pi \varepsilon_0} \int \frac{\lambda(r')}{r^2} \hat{r} \, dl' \]

→ If the charge is smeared out over a surface with charge-per-unit-area \( \sigma \),

\[ dq = \sigma \, da' \quad \rightarrow \quad E(r) = \frac{1}{4\pi \varepsilon_0} \int_S \frac{\sigma(r')}{r^2} \hat{r} \, da' \]

→ If the charge fills a volume with charge-per-unit-volume \( \rho \),

\[ dq = \rho \, d\tau' \quad \rightarrow \quad E(r) = \frac{1}{4\pi \varepsilon_0} \int_V \frac{\rho(r')}{r^2} \hat{r} \, d\tau' \]

@ Please note carefully the meaning of \( \mathbf{r} \) in these formulas.

→ Originally \( \mathbf{r} \) stood for the vector from the source charge \( q_i \) to the field point \( r \).
→ Correspondingly, \( \mathbf{r} \) is the vector from \( dq \) (therefore from \( dl', da' \), or \( d\tau' \)) to the field point \( r \).
2.1.2 Coulomb's Law

Problem 2.7  Find the electric field a distance $z$ from the center of a spherical surface of radius $R$, which carries a uniform surface charge density $\sigma$.

\[
E(r) = \frac{1}{4\pi\epsilon_0} \int \frac{\sigma(r')}{r'^2} \, da'
\]

\[
dq = \sigma \, ds = \sigma R^2 \sin \theta \, d\theta \, d\phi
\]

\[
r^2 = R^2 + z^2 - 2Rz \cos \theta
\]

\[
E(r) = E_z \hat{z};
\]

\[
E_z = |E(r)| \cos \psi
\]

\[
E_z = \frac{1}{4\pi\epsilon_0} \int \frac{\sigma R^2 \sin \theta \, d\theta \, d\phi \sin(z - R \cos \theta)}{(R^2 + z^2 - 2Rz \cos \theta)^{3/2}} \, d\phi = 2\pi
\]

\[
= \frac{1}{4\pi\epsilon_0} (2\pi R^2 \sigma) \int_0^\pi \frac{(z - R \cos \theta) \sin \theta}{(R^2 + z^2 - 2Rz \cos \theta)^{3/2}} \, d\theta
\]

Let $u = \cos \theta$; $du = -\sin \theta \, d\theta$;

\[
\begin{align*}
\theta &= 0 \Rightarrow u = +1 \\
\theta &= \pi \Rightarrow u = -1
\end{align*}
\]

\[
= \frac{1}{4\pi\epsilon_0} (2\pi R^2 \sigma) \left[ \frac{1}{\sqrt{R^2 + z^2 - 2Rzu}} \right]_{-1}^{1} = \frac{1}{4\pi\epsilon_0} \frac{2\pi R^2 \sigma}{z^2} \left\{ \frac{(z - R)}{|z - R|} - \frac{(-z - R)}{|z + R|} \right\}
\]

For $z > R$ (outside the sphere), $E_z = \frac{1}{4\pi\epsilon_0} \frac{4\pi R^2 \sigma}{z^2} = \frac{1}{4\pi\epsilon_0} \frac{\sigma}{z^2}$, so $E = \frac{1}{4\pi\epsilon_0} \frac{q}{z^2} \hat{z}.$

For $z < R$ (inside), $E_z = 0$, so $E = 0$

Look! The integrals involved in computing $E$ may be formidable. ➔ Gauss's Law makes it simple!
2.2.1 Field Lines, Flux, and Gauss's Law

Field lines:

- The magnitude of the field is indicated by the density of the field lines:
  - It's strong near the center where the field lines are close together,
  - It's weak farther out, where they are relatively far apart.
  - The field lines begin on positive charges and end on negative ones;
  - They cannot simply terminate in midair, though they may extend out to infinity;
  - They can never cross.

Field flux: \( \Phi_E = \int_S \mathbf{E} \cdot d\mathbf{a} \) → the "number of field lines" passing through \( S \).

- \( (\mathbf{E} \cdot d\mathbf{a}) \) is proportional to the number of lines passing through the infinitesimal area
- The dot product picks out the component of \( d\mathbf{a} \) along the direction of \( \mathbf{E} \),
- It is only the area in the plane perpendicular to \( \mathbf{E} \).

- The number of field lines is proportional to the magnitude of a charge.

- Therefore, this suggests that
  - "the flux through any closed surface is a measure of the total charge inside"
  - \textbf{This is the essence of Gauss's Law.}
Flux and Gauss's Law

In the case of a point charge $q$ at the origin, the flux of $E$ through a sphere of radius $r$ is

$$\Phi_E = \oint_E E \cdot da = \int \frac{1}{4\pi \epsilon_0} \left( \frac{q}{r^2} \hat{r} \right) \cdot (r^2 \sin \theta d\theta d\phi \hat{r}) = \frac{1}{\epsilon_0} q$$

- The flux through any surface enclosing the charge is $q/\epsilon_0$.

Now suppose a bunch of charges scattered about.
- According to the principle of superposition, the total field is the (vector) sum of all the individual fields:

$$E = \sum_{i=1}^{n} E_i$$

$$\Phi_E = \oint E \cdot da = \sum_{i=1}^{n} \left( \oint E_i \cdot da \right) = \sum_{i=1}^{n} \left( \frac{1}{\epsilon_0} q_i \right) \quad \Rightarrow \quad \oint_{S} E \cdot da = \frac{1}{\epsilon_0} Q_{enc}$$

Gauss’s Law

By applying the divergence theorem:

$$\oint_{S} E \cdot da = \int_{V} (\nabla \cdot E) d\tau$$

$$Q_{enc} = \int_{V} \rho d\tau \quad \Rightarrow \quad \int_{V} (\nabla \cdot E) d\tau = \int_{V} \left( \frac{\rho}{\epsilon_0} \right) d\tau \quad \Rightarrow \quad \nabla \cdot E = \frac{1}{\epsilon_0} \rho$$

Gauss's law in differential form
2.2.2 The Divergence of E

Let’s calculate the divergence of E directly from the Coulomb’s Law of

\[ E(\mathbf{r}) = \frac{1}{4\pi \epsilon_0} \int_{V} \frac{\rho(\mathbf{r}')}{r^2} \hat{\mathbf{r}} \, d\tau' \rightarrow \nabla \cdot E \quad \text{(divergence in terms of } \mathbf{r}) \]

Since the \( \mathbf{r} \)-dependence is contained in \( \mathbf{r} = \mathbf{r} - \mathbf{r}' \), we have

\[
\nabla \cdot E = \frac{1}{4\pi \epsilon_0} \int \nabla \cdot \left( \frac{\hat{\mathbf{r}}}{r^2} \right) \rho(\mathbf{r}') \, d\tau'
\]

\[
\nabla \cdot \left( \frac{\hat{\mathbf{r}}}{r^2} \right) = 4\pi \delta^3(\mathbf{r})
\]

\[
\nabla \cdot E = \frac{1}{4\pi \epsilon_0} \int 4\pi \delta^3(\mathbf{r} - \mathbf{r}') \rho(\mathbf{r}') \, d\tau' = \frac{1}{\epsilon_0} \rho(\mathbf{r})
\]

\[
\nabla \cdot E = \frac{\rho(\mathbf{r})}{\epsilon_0}
\]

\( \rightarrow \text{This is Gauss's law in differential form} \)
2.2.3 Applications of Gauss's Law \( \oint_S \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\varepsilon_0} Q_{\text{enc}} \)

When symmetry permits, it’s the quickest and easiest way of computing electric fields.

**Example 2.2** Find the field outside a uniformly charged solid sphere of radius \( R \) and total charge \( q \). (Remember the Prob. 2.7)

⇒ Draw a spherical surface at radius \( r > R \) ➔ "Gaussian surface"

\[
\oint_S \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\varepsilon_0} Q_{\text{enc}}
\]

\[
\int_S |\mathbf{E}| \, d\mathbf{a} = |\mathbf{E}| \int_S d\mathbf{a} = |\mathbf{E}| 4\pi r^2 = \frac{1}{\varepsilon_0} q
\]

\[
\mathbf{E} = \frac{1}{4\pi \varepsilon_0} \frac{q}{r^2} \hat{\mathbf{r}}
\]

Gauss's law is always true, but it is not always useful.

⇒ Symmetry is crucial to this application of Gauss's law.

1. **Spherical symmetry.** Make your Gaussian surface a concentric sphere.
2. **Cylindrical symmetry.** Make your Gaussian surface a coaxial cylinder.
3. **Plane symmetry.** Use a Gaussian "pillbox," which straddles the surface.
Applications of Gauss's Law \( \oint_S \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\varepsilon_0} Q_{\text{enc}} \)

Example 2.3 A long cylinder carries a charge density that is proportional to the distance from the axis: \( \rho = ks \), for some constant \( k \).

\( \Rightarrow \) The enclosed charge is

\[ Q_{\text{enc}} = \int \rho \, d\tau = \int (ks') (s' \, ds' \, d\phi \, dz) = 2\pi kl \int_0^5 s'^2 \, ds' = \frac{2}{3} \pi kl s^3 \]

Symmetry dictates that \( \mathbf{E} \) must point radially outward,

\[ \oint \mathbf{E} \cdot d\mathbf{a} = \int |\mathbf{E}| \, d\mathbf{a} = |\mathbf{E}| \int d\mathbf{a} = |\mathbf{E}| 2\pi l \]

While the two ends contribute nothing (here \( \mathbf{E} \) is perpendicular to \( d\mathbf{a} \)). Thus,

\[ |\mathbf{E}| 2\pi l = \frac{1}{\varepsilon_0} \frac{2}{3} \pi kl s^3 \rightarrow \mathbf{E} = \frac{1}{3\varepsilon_0} ks \hat{s} \]

Example 2.4 An infinite plane carries a uniform surface charge \( \sigma \).

\( \Rightarrow \) Draw a "Gaussian pillbox," and apply Gauss's law to this surface:

\[ Q_{\text{enc}} = \sigma A \]

By symmetry, \( \mathbf{E} \) points away from the plane (upward for points above, downward for points below)

\[ \oint \mathbf{E} \cdot d\mathbf{a} = 2A |\mathbf{E}| = \frac{1}{\varepsilon_0} \sigma A \rightarrow \mathbf{E} = \frac{\sigma}{2\varepsilon_0} \hat{n} \]
2.2.4 The Curl of $E$

Consider the electric field from a point charge $\mathbf{q}$ at the origin: 

$$\mathbf{E} = \frac{1}{4\pi \varepsilon_0} \frac{q}{r^2} \mathbf{\hat{r}}$$

Now let’s calculate the line integral of this field from some point $\mathbf{a}$ to some other point $\mathbf{b}$: 

$$\int_{a}^{b} \mathbf{E} \cdot d\mathbf{l}$$

In spherical coordinates, $d\mathbf{l} = dr \mathbf{\hat{r}} + r d\theta \mathbf{\hat{\theta}} + r \sin \theta d\phi \mathbf{\hat{\phi}}$

$$\mathbf{E} \cdot d\mathbf{l} = \frac{1}{4\pi \varepsilon_0} \frac{q}{r^2} dr ightarrow \int_{a}^{b} \mathbf{E} \cdot d\mathbf{l} = \frac{1}{4\pi \varepsilon_0} \frac{q}{r^2} dr = -\frac{1}{4\pi \varepsilon_0} \frac{q}{r} \bigg|_{r_a}^{r_b} = \frac{1}{4\pi \varepsilon_0} \left( \frac{q}{r_a} - \frac{q}{r_b} \right)$$

The integral around a closed path is evidently zero (for then $r_a = r_b$):

$$\oint \mathbf{E} \cdot d\mathbf{l} = 0 \Rightarrow \text{Applying Stokes' theorem, } \nabla \times \mathbf{E} = 0$$

If we have many charges, the principle of superposition states that the total field is a vector sum of their individual fields: 

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 + \ldots,$$

$$\nabla \times \mathbf{E} = \nabla \times (\mathbf{E}_1 + \mathbf{E}_2 + \ldots) = (\nabla \times \mathbf{E}_1) + (\nabla \times \mathbf{E}_2) + \ldots = 0$$

$$\nabla \times \mathbf{E} = 0 \Rightarrow \text{For any static charge distribution whatever}$$