Chapter 4. Electric Fields in Matter

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4.1.1 Dielectrics

All charges are attached to specific atoms or molecules.  
→ They can move a bit within the atom or molecule.  
→ Such microscopic displacements account for the characteristic behavior of dielectric materials.

There are actually two principal mechanisms by which electric fields can distort the charge distribution of a dielectric atom or molecule:  
→ stretching and rotating.

\[ E_{\text{applied}} = 0 \]  
\[ E_{\text{applied}} \neq 0 \]
4.1.2 Induced Dipoles

What happens to a neutral atom when it is placed in an electric field $E$?

The nucleus is pushed in the direction of the field, and the electrons the opposite way. The two opposing forces reach a balance, leaving the atom polarized. The atom now has a tiny dipole moment $\mathbf{p}$, which points in the same direction as $E$. Typically, this induced dipole moment is approximately proportional to the field (as long as the latter is not too strong):

$$\mathbf{p} = \alpha \mathbf{E}$$

: $\alpha$ is called atomic polarizability

Atomic Polarizabilities ($\alpha / 4\pi \epsilon_0$, in units of $10^{-30} \text{ m}^3$)

<table>
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<th>H</th>
<th>He</th>
<th>Li</th>
<th>Be</th>
<th>C</th>
<th>Ne</th>
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<th>Ar</th>
<th>K</th>
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<td>5.60</td>
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<td>24.1</td>
<td>1.64</td>
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**Example 4.1** An atom consists of a point nucleus (+$q$) surrounded by a uniformly charged spherical cloud (-$q$) of radius $a$. Calculate the atomic polarizability of such an atom.

It is reasonable to assume that the electron cloud retains its spherical shape. Equilibrium occurs when the nucleus is displaced a distance $d$ from the center of the sphere and pulling by the internal field produced by the electron cloud, $E_e$.

At equilibrium, $E = E_e$

$$E = E_e = \frac{1}{4\pi \epsilon_0} \frac{q d}{a^3} \quad \rightarrow \quad p = q d = (4\pi \epsilon_0 a^3) E \quad \rightarrow \quad \alpha = 4\pi \epsilon_0 a^3 = 3\epsilon_0 v$$
According to quantum mechanics, the electron cloud for a hydrogen atom in the ground state has a charge density.

\[
\rho(r) = \frac{q}{\pi a^3} e^{-2r/a}
\]

(not uniformly charged case)

where \( q \) is the charge of the electron and \( a \) is the Bohr radius.

Find the atomic polarizability of such an atom.

First find the field, at radius \( r \), using Gauss’ law:

\[
\frac{\int \mathbf{E} \cdot d\mathbf{A}}{\epsilon_0} = \frac{Q_{\text{enc}}}{\epsilon_0} \quad \Rightarrow \quad \mathbf{E} = \frac{1}{4\pi \epsilon_0} \frac{1}{r^2} Q_{\text{enc}}
\]

\[
Q_{\text{enc}} = \int_0^r \rho \, d\tau = \frac{4\pi q}{\pi a^3} \int_0^r e^{-2r/a} \tau^2 \, d\tau = \frac{4q}{a^3} \left[ -\frac{a}{2} e^{-2r/a} \left( r^2 + ar + \frac{a^2}{2} \right) \right]_0^r = -\frac{2q}{a^2} \left[ e^{-2r/a} \left( r^2 + ar + \frac{a^2}{2} \right) - \frac{a^2}{2} \right]
\]

So the field of the electron cloud is

\[
E_c = \frac{1}{4\pi \epsilon_0} \frac{q}{r^2} \left[ 1 - e^{-2r/a} \left( 1 + 2\frac{r}{a} + 2\frac{r^2}{a^2} \right) \right]
\]

The proton will be shifted from \( r = 0 \) to the point \( d \) where \( E_c = E \)

\[
E = \frac{1}{4\pi \epsilon_0} \frac{q}{d^2} \left[ 1 - e^{-2d/a} \left( 1 + 2\frac{d}{a} + 2\frac{d^2}{a^2} \right) \right]
\]

\[
1 - e^{-2d/a} \left( 1 + 2\frac{d}{a} + 2\frac{d^2}{a^2} \right) = 1 - \left( 1 - 2\frac{d}{a} + \frac{4}{3} \left( \frac{d}{a} \right)^3 + \cdots \right) \left( 1 + 2\frac{d}{a} + 2\frac{d^2}{a^2} \right) = \frac{4}{3} \left( \frac{d}{a} \right)^3 + \text{higher order terms}
\]

\[
E = \frac{1}{4\pi \epsilon_0} \frac{q}{d^2} \left( \frac{4}{3} \frac{d^3}{a^3} \right) = \frac{1}{4\pi \epsilon_0} \frac{4}{3a^3} (qd) = \frac{1}{3\pi \epsilon_0 a^3 p} \quad \Rightarrow \quad \alpha = \frac{3\pi \epsilon_0 a^3}{4\pi \epsilon_0} = 3\epsilon_0 \nu
\]
Polarizability tensor

Carbon dioxide (CO$_2$), for instance, when the field is at some angle to the axis, you must resolve it into parallel and perpendicular components, and multiply each by the pertinent polarizability:

\[ \mathbf{p} = \alpha_\perp \mathbf{E}_\perp + \alpha_\parallel \mathbf{E}_\parallel \]

\[ \alpha_\parallel \neq \alpha_\perp \]

A most general linear relation between $E$ and $p$ is

\[
\begin{align*}
 p_x &= \alpha_{xx} E_x + \alpha_{xy} E_y + \alpha_{xz} E_z \\
 p_y &= \alpha_{yx} E_x + \alpha_{yy} E_y + \alpha_{yz} E_z \\
 p_z &= \alpha_{zx} E_x + \alpha_{zy} E_y + \alpha_{zz} E_z
\end{align*}
\]

\[ \quad \quad \quad \rightarrow \quad \quad \quad \mathbf{p} = \overrightarrow{\alpha \mathbf{E}} \]

The set of nine constants $\alpha_{ij}$ \textbf{\Rightarrow polarizability tensor}

By choose "principal" axes, we can leave just three nonzero polarizabilities: $\alpha_{xx}$, $\alpha_{yy}$, and $\alpha_{zz}$. 
4.1.3 Alignment of Polar Molecules

What happens when polar molecules are placed in an electric field?

**Polar molecules**: molecules having built-in dipole moments

(H$_2$O, for example)

There is a **torque**:

\[
N = (r_+ \times F_+) + (r_- \times F_-)
\]

\[
= [(d/2) \times (qE)] + [(-d/2) \times (-qE)] = qd \times E.
\]

\(N = p \times E\) \(\Rightarrow\) A polar molecule that is free to rotate will swing around until it points in the direction of the applied field.

**If the field is nonuniform**, so that \(F_+\) does not exactly balance \(F_-\), there will be a net force on the dipole, in addition to the torque.

\[
F = F_+ + F_- = q(E_+ - E_-) = q(\Delta E)
\]

Assuming the dipole is very short, \(\Delta E = \nabla E \cdot d = (d \cdot \nabla)E\)

\[
F = (p \cdot \nabla)E
\]

\[
N = (p \times E) + (r \times F)
\]
Energy of an ideal dipole $p$ in $E$

**Problem 4.7** Show that the energy of an ideal dipole $p$ in an electric field $E$ is given by

$$U = -p \cdot E$$

The energy of a point charge $Q$ is $U = QV(r)$.

$\Rightarrow$ For a physical dipole, with $-q$ at $r$ and $+q$ at $r + d$, the energy is

$$U = qV(r + d) - qV(r) = q \left[ - \int_r^{r+d} E \cdot dl \right]$$

$\Rightarrow$ For an ideal dipole the integral reduces to $E \cdot d$,

$$U = -qE \cdot d = -p \cdot E$$

**Problem 4.9** A dipole $p$ is a distance $r$ from a point charge $q$, and oriented so that $p$ makes an $\theta$ with the vector $r$ from $q$ to $p$.

What is the force on $p$?

$$F = (p \cdot \nabla)E \leftarrow E = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{r} = \frac{q}{4\pi\epsilon_0} \frac{x\hat{x} + y\hat{y} + z\hat{z}}{(x^2 + y^2 + z^2)^{3/2}}$$

$$F_x = \left( p_x \frac{\partial}{\partial x} + p_y \frac{\partial}{\partial y} + p_z \frac{\partial}{\partial z} \right) \frac{q}{4\pi\epsilon_0} \frac{x}{(x^2 + y^2 + z^2)^{3/2}} = \frac{q}{4\pi\epsilon_0} \left[ \frac{p}{r^3} - \frac{3r(p \cdot r)}{r^5} \right]_x$$

$$F = \frac{1}{4\pi\epsilon_0} \frac{q}{r^3} [p - 3(p \cdot \hat{r}) \hat{r}]$$
4.1.4 Polarization

What happens to a piece of dielectric material when it is placed in an electric field?

A lot of little dipoles pointing along the direction of the field. The material becomes polarized.

\[
P = \lim_{{\Delta \nu \to 0}} \left[ \sum_i p_i / \Delta \nu \right] (C/m^2)
\]

: Dipole moment per unit volume

This Polarization vector will displace the electric strength in a dielectric medium:

\[
D \equiv \epsilon_0 E + P
\]

: D is known as the electric displacement

\[
P = \epsilon_0 \chi_e E
\]

: For linear dielectrics

\[
D = \epsilon_0 E + P = \epsilon_0 E + \epsilon_0 \chi_e E = \epsilon_0 (1 + \chi_e)E
\]

\[
D = \epsilon E \quad \epsilon \equiv \epsilon_0 (1 + \chi_e) \quad \text{: Permittivity}
\]

\[
\epsilon_r \equiv 1 + \chi_e = \frac{\epsilon}{\epsilon_0} \quad \text{: Relative permittivity}
\]