Chapter 9. Electromagnetic Waves

9.1 Waves in One Dimension
  9.1.1 The Wave Equation
  9.1.2 Sinusoidal Waves
  9.1.3 Boundary Conditions: Reflection and Transmission
  9.1.4 Polarization

9.2 Electromagnetic Waves in Vacuum
  9.2.1 The Wave Equation for \( \mathbf{E} \) and \( \mathbf{B} \)
  9.2.2 Monochromatic Plane Waves
  9.2.3 Energy and Momentum in Electromagnetic Waves

9.3 Electromagnetic Waves in Matter
  9.3.1 Propagation in Linear Media
  9.3.2 Reflection and Transmission at Normal Incidence
  9.3.3 Reflection and Transmission at Oblique Incidence

9.4 Absorption and Dispersion
  9.4.1 Electromagnetic Waves in Conductors
  9.4.2 Reflection at a Conducting Surface
  9.4.3 The Frequency Dependence of Permittivity

9.5 Guided Waves
  9.5.1 Wave Guides
  9.5.2 TE Waves in a Rectangular Wave Guide
  9.5.3 The Coaxial Transmission Line
9.2 Electromagnetic waves in Vacuum

9.2.1 The Wave Equation for E and B

In Vacuum, no free charges and no currents \( \rho = 0, \ J = 0, \ q = 0, \ I = 0 \)

\[
\begin{align*}
\nabla \cdot \mathbf{E} &= 0 \\
\nabla \cdot \mathbf{B} &= 0 \\
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\
\nabla \times \mathbf{B} &= \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}
\end{align*}
\]

Let's derive the wave equation for \( \mathbf{E} \) and \( \mathbf{B} \) from the curl equations.

\[
\begin{align*}
\nabla \times (\nabla \times \mathbf{E}) &= \nabla (\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = \nabla \times \left( -\frac{\partial \mathbf{B}}{\partial t} \right) = -\frac{\partial}{\partial t} \left( \nabla \times \mathbf{B} \right) = -\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} \\
\nabla \times (\nabla \times \mathbf{B}) &= \nabla (\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B} = \nabla \times \left( \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) = \mu_0 \varepsilon_0 \frac{\partial}{\partial t} \left( \nabla \times \mathbf{E} \right) = -\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2}
\end{align*}
\]

Each Cartesian component of \( \mathbf{E} \) and \( \mathbf{B} \) satisfies

\[
\nabla^2 f = \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2}, \quad v = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} = 3.00 \times 10^8 \text{ m/s}
\]

Notice the crucial role played by Maxwell’s contribution to Ampere’s law \((\mu_0 \varepsilon_0 \partial \mathbf{E}/\partial t)\);

→ Without it, the wave equation would not emerge,
→ there would be no electromagnetic theory of light.
9.2.2 Monochromatic Plane Waves

Sinusoidal waves with a frequency $\omega$ are called monochromatic.

Sinusoidal waves traveling in the z direction and have no x or y dependence, are called plane waves, because the fields are uniform over every plane perpendicular to the direction of propagation.

$$\tilde{E}(z, t) = \tilde{E}_0 e^{i(kz - \omega t)}$$

$$\tilde{B}(z, t) = \tilde{B}_0 e^{i(kz - \omega t)}$$

### The Electromagnetic Spectrum

<table>
<thead>
<tr>
<th>Frequency (Hz)</th>
<th>Type</th>
<th>Wavelength (m)</th>
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<tr>
<td>$10^{22}$</td>
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<tr>
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<td>AM</td>
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### The Visible Range

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<th>Wavelength (m)</th>
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<td>blue</td>
<td>$4.6 \times 10^{-7}$</td>
</tr>
<tr>
<td>$5.6 \times 10^{14}$</td>
<td>green</td>
<td>$5.4 \times 10^{-7}$</td>
</tr>
<tr>
<td>$5.1 \times 10^{14}$</td>
<td>yellow</td>
<td>$5.9 \times 10^{-7}$</td>
</tr>
<tr>
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<td>orange</td>
<td>$6.1 \times 10^{-7}$</td>
</tr>
<tr>
<td>$3.9 \times 10^{14}$</td>
<td>longest visible red</td>
<td>$7.6 \times 10^{-7}$</td>
</tr>
<tr>
<td>$3.0 \times 10^{14}$</td>
<td>near infrared</td>
<td>$1.0 \times 10^{-6}$</td>
</tr>
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Monochromatic Plane Waves

\[ \vec{E}(z, t) = \vec{E}_0 e^{i(kz - \omega t)} \]
\[ \vec{B}(z, t) = \vec{B}_0 e^{i(kz - \omega t)} \]

since \( \nabla \cdot \vec{E} = 0 \) and \( \nabla \cdot \vec{B} = 0 \)

\[ (\vec{E}_0)_z = (\vec{B}_0)_z = 0 \]

⇒ Electromagnetic waves are transverse:

→ the electric and magnetic fields are perpendicular to the direction of propagation

\[ \vec{E}(z, t) = \vec{E}_0 e^{i(kz - \omega t)} \]
\[ \vec{B}(z, t) = \vec{B}_0 e^{i(kz - \omega t)} \]

Faraday’s law, \( \nabla \times \vec{E} = -\partial \vec{B}/\partial t \)

\[ -k(\vec{E}_0)_y = \omega (\vec{B}_0)_x \quad k(\vec{E}_0)_x = \omega (\vec{B}_0)_y \]

\[ \vec{B}_0 = \frac{k}{\omega} (\hat{z} \times \vec{E}_0) \]

Evidently, \( \vec{E} \) and \( \vec{B} \) are in phase and mutually perpendicular;

their (real) amplitudes are related by

\[ B_0 = \frac{k}{\omega} E_0 = \frac{1}{c} E_0 \]

Example 9.2 If \( \vec{E} \) points in the \( x \) direction, then \( \vec{B} \) points in the \( y \) direction,

\[ \vec{E}(z, t) = \vec{E}_0 e^{i(kz - \omega t)} \hat{x} \quad \vec{B}(z, t) = \frac{1}{c} \vec{E}_0 e^{i(kz - \omega t)} \hat{y} \]

Or, taking the real part,

\[ \vec{E}(z, t) = E_0 \cos(kz - \omega t + \delta) \hat{x} \quad \vec{B}(z, t) = \frac{1}{c} E_0 \cos(kz - \omega t + \delta) \hat{y} \]

The monochromatic plane wave as a whole is said to be polarized in the \( x \) direction
(by convention, we use the direction of \( \vec{E} \) to specify the polarization of an electromagnetic wave).
Monochromatic Plane Waves

We can easily generalize to monochromatic plane waves traveling in an arbitrary direction.

\[
\tilde{E}(z, t) = \tilde{E}_0 e^{i(kz - \omega t)}
\]
\[
\tilde{B}(z, t) = \tilde{B}_0 e^{i(kz - \omega t)}
\]

\[
\tilde{E}(r, t) = \tilde{E}_0 e^{i(k \cdot r - \omega t)} \hat{n}
\]
\[
\tilde{B}(r, t) = \frac{1}{c} \tilde{E}_0 e^{i(k \cdot r - \omega t)} (\hat{k} \times \hat{n}) = \frac{1}{c} \hat{k} \times \tilde{E}
\]

**k** \(\rightarrow\) propagation (or wave) vector, pointing in the direction of propagation, whose magnitude is the wave number \(k\).

\(\hat{n}\) is the polarization vector.

Because \(E\) is transverse \(\rightarrow\)

\[
\hat{n} \cdot \hat{k} = 0
\]

The actual (real) electric and magnetic fields in a monochromatic plane wave with propagation vector \(k\) and polarization \(n\) are

\[
E(r, t) = E_0 \cos (k \cdot r - \omega t + \delta) \hat{n}
\]
\[
B(r, t) = \frac{1}{c} E_0 \cos (k \cdot r - \omega t + \delta)(\hat{k} \times \hat{n})
\]
Summary of Important Properties of Electromagnetic Waves

- The solutions (plane wave) of Maxwell’s equations are wave-like with both $E$ and $B$ satisfying a wave equation.

$$E_y = E_0 \cos(kx - \omega t)$$

$$B_z = B_0 \cos(kx - \omega t)$$

- Electromagnetic waves travel through empty space with the speed of light $c = 1/(\mu_0 \varepsilon_0)^{1/2}$.

- The plane wave as represented by above is said to be linearly polarized because the electric vector is always along $y$-axis and, similarly, the magnetic vector is always along $z$-axis.
The components of the electric and magnetic fields of plane EM waves are perpendicular to each other and perpendicular to the direction of wave propagation. The latter property says that EM waves are transverse waves.

The magnitudes of $E$ and $B$ in empty space are related by $E/B = c$. 

$$\frac{E_o}{B_o} = \frac{E}{B} = \frac{\omega}{k} = c$$

The electric and magnetic waves are interdependent; neither can exist without the other. Physically, an electric field varying in time produces a magnetic field varying in space and time; this changing magnetic field produces an electric field varying in space and time and so on. This mutual generation of electric and magnetic fields result in the propagation of the EM waves.
9.2.3 Energy and Momentum in Electromagnetic Waves

Energy per unit volume stored in electromagnetic fields $\rightarrow \quad u = \frac{1}{2} \left( \varepsilon_0 E^2 + \frac{1}{\mu_0} B^2 \right)$

In the case of a monochromatic plane wave,

$\mathbf{E}(z, t) = E_0 \cos(kz - \omega t + \delta) \hat{x}$

$\mathbf{B}(z, t) = \frac{1}{c} E_0 \cos(kz - \omega t + \delta) \hat{y}$

$\mathbf{E}^2 = \frac{1}{c^2} E^2 = \mu_0 \varepsilon_0 E^2$ $\rightarrow$ $u = \varepsilon_0 E^2 = \varepsilon_0 E_0^2 \cos^2 (kz - \omega t + \delta)$

$\Rightarrow$ Electric and magnetic contributions are equal

Energy flux density (energy per unit area, per unit time; Poynting vector)

For monochromatic plane waves propagating in the $z$ direction,

$\mathbf{S} = \frac{1}{\mu_0} (\mathbf{E} \times \mathbf{B})$ $\rightarrow$ $\mathbf{S} = c \varepsilon_0 E_0^2 \cos^2 (kz - \omega t + \delta) \hat{z} = cu \hat{z}$

Momentum density stored in the fields $\rightarrow \quad \mathbf{g} = \varepsilon_0 \mu_0 \mathbf{S} = \frac{1}{c^2} \mathbf{S}$

For monochromatic plane waves $\rightarrow \quad \mathbf{g} = \frac{1}{c^2} \mathbf{S} = \frac{1}{c} \varepsilon_0 E_0^2 \cos^2 (kz - \omega t + \delta) \hat{z} = \frac{1}{c} u \hat{z}$

Intensity (time average power per unit area transported by an electromagnetic wave)

$I \equiv \langle S \rangle = \frac{1}{2} c \varepsilon_0 E_0^2 \rightarrow \text{brackets, } \langle \cdot \rangle, \text{ to denote the (time) average over a complete cycle (or many cycles)}$

$\langle u \rangle = \frac{1}{2} \varepsilon_0 E_0^2 \quad \langle S \rangle = \frac{1}{2} c \varepsilon_0 E_0^2 \hat{z}$

$\langle \mathbf{g} \rangle = \frac{1}{2c} \varepsilon_0 E_0^2 \hat{z}$
Problem 9.11 In the complex notation there is a clever device for finding the time average of a product. Suppose

\[ f(\mathbf{r}, t) = A \cos(\mathbf{k} \cdot \mathbf{r} - \omega t + \delta_a) \quad g(\mathbf{r}, t) = B \cos(\mathbf{k} \cdot \mathbf{r} - \omega t + \delta_b) \]

Show that \( \langle fg \rangle = (1/2) \text{Re}(\tilde{f} \tilde{g}^*) \)

\[
\langle fg \rangle = \frac{1}{T} \int_0^T a \cos(\mathbf{k} \cdot \mathbf{r} - \omega t + \delta_a) b \cos(\mathbf{k} \cdot \mathbf{r} - \omega t + \delta_b) \, dt
\]

\[
= \frac{ab}{2T} \int_0^T \left[ \cos(2\mathbf{k} \cdot \mathbf{r} - 2\omega t + \delta_a + \delta_b) + \cos(\delta_a - \delta_b) \right] \, dt
\]

\[
= \frac{ab}{2T} \cos(\delta_a - \delta_b)T = \frac{1}{2} ab \cos(\delta_a - \delta_b)
\]

Meanwhile, in the complex notation:

\[
\tilde{f} = \tilde{a} e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} \quad \tilde{g} = \tilde{b} e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} \quad \text{where} \quad \tilde{a} = ae^{i\delta_a} \quad \tilde{b} = be^{i\delta_b}
\]

\[
\frac{1}{2} \tilde{f} \tilde{g}^* = \frac{1}{2} \tilde{a} e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} \overline{\tilde{b}}^* e^{-i(\mathbf{k} \cdot \mathbf{r} - \omega t)}
\]

\[
= \frac{1}{2} \tilde{a} \overline{\tilde{b}}^* = \frac{1}{2} ab e^{i(\delta_a - \delta_b)}
\]

\[
\text{Re} \left( \frac{1}{2} \tilde{f} \tilde{g}^* \right) = \frac{1}{2} ab \cos(\delta_a - \delta_b) = \langle fg \rangle
\]

\[
\begin{align*}
\langle u \rangle & = \frac{1}{2} \left( \epsilon_0 E^2 + \frac{1}{\mu_0} B^2 \right) \\
\langle S \rangle & = \frac{1}{\mu_0} (\mathbf{E} \times \mathbf{B})
\end{align*}
\]

Monochromatic plane waves

\[
\begin{align*}
\langle u \rangle & = \frac{1}{2} \epsilon_0 E_0^2 \\
\langle S \rangle & = \frac{1}{2c\epsilon_0} E_0^2 \mathbf{\hat{z}}
\end{align*}
\]
Maxwell stress tensor for a monochromatic plane wave

Problem 9.12  Find all elements of the Maxwell stress tensor for a monochromatic plane wave traveling in the \( z \) direction and linearly polarized in the \( x \) direction.

\[
\mathbf{E}(z, t) = E_0 \cos(kz - \omega t + \delta) \hat{x}
\]
\[
\mathbf{B}(z, t) = \frac{1}{c} E_0 \cos(kz - \omega t + \delta) \hat{y}
\]

\[
T_{ij} = \varepsilon_0 \left( E_i E_j - \frac{1}{2} \delta_{ij} E^2 \right) + \frac{1}{\mu_0} \left( B_i B_j - \frac{1}{2} \delta_{ij} B^2 \right)
\]

\( \mathbf{E} \) has only an \( x \) component, and \( \mathbf{B} \) only a \( y \) component \( \rightarrow \) all the “off-diagonal” \( (i \neq j) \) terms are zero.

For the “diagonal” elements:

\[
T_{xx} = \varepsilon_0 \left( E_x E_x - \frac{1}{2} E^2 \right) + \frac{1}{\mu_0} \left( -\frac{1}{2} B^2 \right) = \frac{1}{2} \left( \varepsilon_0 E^2 - \frac{1}{\mu_0} B^2 \right) = 0
\]
\[
T_{yy} = \varepsilon_0 \left( -\frac{1}{2} E^2 \right) + \frac{1}{\mu_0} \left( B_y B_y - \frac{1}{2} B^2 \right) = \frac{1}{2} \left( -\varepsilon_0 E^2 + \frac{1}{\mu_0} B^2 \right) = 0
\]
\[
T_{zz} = \varepsilon_0 \left( -\frac{1}{2} E^2 \right) + \frac{1}{\mu_0} \left( -\frac{1}{2} B^2 \right) = u = -\varepsilon_0 E_0^2 \cos^2(kz - \omega t + \delta)
\]

The momentum of these fields is in the \( z \) direction, and it is being transported in the \( z \) direction since \( \mathbf{g} = \varepsilon_0 \mu_0 \mathbf{S} \).

Also note that the flow of momentum (momentum per unit area, unit time) carried by EM fields \( \rightarrow \) \( -\mathbf{T} \).

\( \rightarrow \) Therefore, it does make sense that \( T_{zz} \) should be the only nonzero element in \( T_{ij} \).