Chapter 4. Electric Fields in Matter

4 Electric Fields in Matter

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4.1.2 Induced Dipoles

What happens to a neutral atom when it is placed in an electric field $E$?

The atom now has a tiny dipole moment $p$, in the same direction as $E$.

Typically, $P$ is approximately proportional to the field.

$p = \alpha E$

: $\alpha$ is called atomic polarizability

Atomic Polarizabilities ($\alpha/4\pi \varepsilon_0$, in units of $10^{-30}$ m$^3$)

<table>
<thead>
<tr>
<th></th>
<th>H</th>
<th>He</th>
<th>Li</th>
<th>Be</th>
<th>C</th>
<th>Ne</th>
<th>Na</th>
<th>Ar</th>
<th>K</th>
<th>Cs</th>
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<tr>
<td>$\alpha$</td>
<td>0.667</td>
<td>0.205</td>
<td>24.3</td>
<td>5.60</td>
<td>1.76</td>
<td>0.396</td>
<td>24.1</td>
<td>1.64</td>
<td>43.4</td>
<td>59.6</td>
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Example 4.1  An atom consists of a point nucleus (+$q$) surrounded by a uniformly charged spherical cloud (-$q$) of radius $a$. Calculate the atomic polarizability of such an atom.

It is reasonable to assume that the electron cloud retains its spherical shape. Equilibrium occurs when the nucleus is displaced a distance $d$ from the center of the sphere and pulling by the internal field produced by the electron cloud, $E_e$:

At equilibrium, $E = E_e$

$$E = E_e = \frac{1}{4\pi \varepsilon_0} \frac{qd}{a^3} \rightarrow p = qd = (4\pi \varepsilon_0 a^3)E \rightarrow \alpha = 4\pi \varepsilon_0 a^3 = 3\varepsilon_0 v$$
Polarizability tensor

Carbon dioxide (CO₂), for instance, *when the field is at some angle to the axis*, you must resolve it into parallel and perpendicular components, and multiply each by the pertinent polarizability:

\[
p = \alpha_\perp E_\perp + \alpha_\parallel E_\parallel
\]

\[
\alpha_\parallel \neq \alpha_\perp
\]

A most general linear relation between \( E \) and \( p \) is

\[
\begin{align*}
p_x &= \alpha_{xx} E_x + \alpha_{xy} E_y + \alpha_{xz} E_z \\
p_y &= \alpha_{yx} E_x + \alpha_{yy} E_y + \alpha_{yz} E_z \\
p_z &= \alpha_{zx} E_x + \alpha_{zy} E_y + \alpha_{zz} E_z
\end{align*}
\]

\[
p = \alpha \vec{E}
\]

The set of nine constants \( \alpha_{ij} \) \( \Rightarrow \) polarizability tensor

By choose "principal" axes, we can leave just three nonzero polarizabilities: \( \alpha_{xx} \), \( \alpha_{yy} \), and \( \alpha_{zz} \).
What happens to a piece of dielectric material when it is placed in an electric field?

A lot of little dipoles pointing along the direction of the field. The material becomes polarized.

\[
P = \lim_{\Delta \nu \to 0} \left[ \sum_i p_i / \Delta \nu \right] \quad \text{(C/m}^2) \quad : \text{Total dipole moment per unit volume}
\]

Polarization

This Polarization vector will displace the electric strength in a dielectric medium:

\[
\mathbf{D} \equiv \epsilon_0 \mathbf{E} + \mathbf{P} \quad : \text{D is known as the electric displacement}
\]

\[
\mathbf{P} = \epsilon_0 \chi_e \mathbf{E} \quad : \text{For linear dielectrics}
\]

\[
\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} = \epsilon_0 \mathbf{E} + \epsilon_0 \chi_e \mathbf{E} = \epsilon_0 (1 + \chi_e) \mathbf{E}
\]

\[
\mathbf{D} = \epsilon \mathbf{E} \quad \epsilon \equiv \epsilon_0 (1 + \chi_e) \quad : \text{Permittivity}
\]

\[
\epsilon_r \equiv 1 + \chi_e = \frac{\epsilon}{\epsilon_0} \quad : \text{Relative permittivity}
\]
4.2 The Field of a Polarized Object

What is the field produced by \( P \) in a polarized material?
(not the field that may have caused the polarization, but the field the polarization itself causes)

4.2.1 Bound Charges

The polarization \( P \) can produce *bound volume and surface charges*.

\[
\rho_b = -\nabla \cdot P
\]

\[
\sigma_b = P \cdot \hat{n}
\]
4.2.1 Bound Charges

For a single dipole \( \mathbf{p} \), the potential is,

\[
V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{\hat{r} \cdot \mathbf{p}}{r^2}
\]

A dipole moment in each volume element \( d\tau' \):

\[
\mathbf{p} = \mathbf{P} d\tau'
\]

\[
V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_{V} \frac{\hat{r} \cdot \mathbf{P}(\mathbf{r}')}{r^2} d\tau'
\]

\[
\nabla' \left( \frac{1}{r} \right) = \frac{\hat{r}}{r^2} \quad \Leftarrow \text{the differentiation is with respect to the source coordinates (r')}
\]

\[
V = \frac{1}{4\pi\epsilon_0} \int_{V} \mathbf{P} \cdot \nabla' \left( \frac{1}{r} \right) d\tau'
\]

Integrating by parts,

\[
V = \frac{1}{4\pi\epsilon_0} \left[ \int_{V} \nabla' \cdot \left( \frac{\mathbf{P}}{r} \right) d\tau' - \int_{V} \frac{1}{r} (\nabla' \cdot \mathbf{P}) d\tau' \right]
\]

Using the divergence theorem,

\[
V = \frac{1}{4\pi\epsilon_0} \int_{S} \frac{\sigma_b}{r} d\mathbf{a}' - \frac{1}{4\pi\epsilon_0} \int_{V} \frac{1}{r} (\nabla' \cdot \mathbf{P}) d\tau'
\]

\[
\begin{align*}
V(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0} \int_{S} \frac{\sigma_b}{r} d\mathbf{a}' + \frac{1}{4\pi\epsilon_0} \int_{V} \frac{\rho_b}{r} d\tau' \\
\rho_b &\equiv -\nabla \cdot \mathbf{P} : \text{Volume charge density} \\
\sigma_b &\equiv \mathbf{P} \cdot \hat{n} : \text{Surface charge density}
\end{align*}
\]
4.2.2 Physical Interpretation of Bound Charges

If the ends have been sliced off perpendicularly, the net bound charge in a given volume is equal and opposite to the amount that has been pushed out through the surface:

\[ \sigma_b = \frac{q}{A} = P \quad \Rightarrow \quad \sigma_b \equiv \mathbf{P} \cdot \hat{n} \]

For an oblique cut, the area is given by:

\[ A = A_{\text{end}} \cos \theta, \]

The net bound charge is:

\[ \rho_b = \frac{q}{A_{\text{end}}} = P \cos \theta \quad \Rightarrow \quad \rho_b \equiv \mathbf{P} \cdot \hat{n} \]

If the polarization is nonuniform, we get accumulations of bound charge within the material. The net bound charge \( \int \rho_b \, d\tau \) in a given volume is equal and opposite to the amount that has been pushed out through the surface:

\[ \int_{V} \rho_b \, d\tau = -\oint_{S} \mathbf{P} \cdot d\mathbf{a} = -\int_{V} (\nabla \cdot \mathbf{P}) \, d\tau \quad \Rightarrow \quad \rho_b \equiv -\nabla \cdot \mathbf{P} \]
Electric field \((E)\) produced by a uniformly polarized \((P)\) sphere

The field inside a uniformly charged sphere is

\[
\oint E \cdot d\alpha = E \cdot 4\pi r^2 = \frac{1}{\varepsilon_0} Q_{\text{enc}} = \frac{1}{\varepsilon_0} \frac{4}{3} \pi r^3 \rho. \quad \longrightarrow \quad E = \frac{1}{3\varepsilon_0} \rho \hat{r}
\]

A uniformly polarized sphere can be regarded as the case when two spheres are overlapped:

One is positive and the other is negative in charge.

The field in the region of overlap between two uniformly charged spheres is

\[
E = \frac{\rho}{3\varepsilon_0} (r_+ - r_-) = \frac{\rho}{3\varepsilon_0} d \quad \longrightarrow \quad E = -\frac{1}{4\pi \varepsilon_0} \frac{qd}{R^3}
\]

\[
p = qd = \left(\frac{4}{3} \pi R^3\right) P \quad \longrightarrow \quad E = -\frac{1}{3\varepsilon_0} P
\]
4.3 The Electric Displacement

4.3.1 Gauss's Law in the Presence of Dielectrics

The effect of polarization (P) is to produce accumulations of bound charge:

\[ \rho_b \equiv -\nabla \cdot P \quad : \text{within the dielectric} \quad \sigma_b \equiv P \cdot \hat{n} \quad : \text{on the surface} \]

If any free charge exists within a dielectric, therefore, the total charge density can be written as

\[ \rho = \rho_b + \rho_f \quad (\rho_f : \text{free charge density}) \]

\[ \Rightarrow \text{Gauss's law reads} \]

\[ \varepsilon_0 \nabla \cdot E = \rho = \rho_b + \rho_f = -\nabla \cdot P + \rho_f \quad \Rightarrow \nabla \cdot (\varepsilon_0 E + P) = \rho_f \]

\[ D = \varepsilon_0 E + P \quad (C/m^2) \quad \Rightarrow \text{Electric displacement} \]

\[ \nabla \cdot D = \rho_f \]

\[ \oint D \cdot da = Q_{f_{\text{enc}}} \]

\[ Q_{f_{\text{enc}}} \text{ denotes the total free charge enclosed in the volume.} \]

\[ \text{In a typical problem, we know } \rho_f, \text{ but we do not (initially) know } \rho_b \]

\[ \text{Whenever the requisite symmetry is present, we can immediately calculate } D \text{ by the Gauss's law method, because it makes reference only to free charges.} \]
Problem 4.15  
A thick spherical shell (inner radius \( a \), outer radius \( b \)) is made of dielectric material with a "frozen-in" polarization \( P(r) \). (No free charge inside)

Find the electric field in all three regions by two different methods:

(a) Identify all the bound charge, and use Gauss's law to calculate the field it produces.

\[
\rho_b = -\nabla \cdot P = -\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{k}{r} \right) = -\frac{k}{r^2}; \quad \quad \quad P(r) = \frac{k}{r} \hat{r}
\]

\[
\sigma_b = P \cdot \hat{n} = \begin{cases} 
+P \cdot \hat{r} = k/b \quad \text{(at } r = b\text{)}, \\
-P \cdot \hat{r} = -k/a \quad \text{(at } r = a\text{)}. 
\end{cases}
\]

For \( r < a \), \( Q_{\text{enc}} = 0 \), so Gauss's law \( \Rightarrow \) \( E = 0 \).

For \( r > b \), \( Q_{\text{tot}} = \int_S \sigma_b \, da + \int_V \rho_b \, d\tau = \int_S P \cdot \, da - \int_V \nabla \cdot P \, d\tau \)

(divergence theorem)

\[
\int_S P \cdot \, da = \int_V \nabla \cdot P \, d\tau, \quad \text{so} \quad Q_{\text{enc}} = 0 \quad \text{Gauss's law } \Rightarrow \quad E = 0
\]

For \( a < r < b \), \( Q_{\text{enc}} = \left( \frac{-k}{a} \right) \left( 4\pi a^2 \right) + \int_a^r \left( \frac{-k}{r^2} \right) 4\pi r^2 \, d\tau = -4\pi ka - 4\pi k(r-a) = -4\pi kr; \quad \text{so} \)

\[
E = -(k/\varepsilon_0 r) \hat{r}.
\]

(b) Use the equations of electric displacement \( D \): \( D = \varepsilon_0 E + P \)

\[
\oint D \cdot da = Q_{\text{fenc}} = 0 \quad \Rightarrow \quad D = 0 \quad \text{everywhere.}
\]

\[
E = 0 \quad \text{(for } r < a \text{ and } r > b\text{)}; \quad \quad \quad E = -(k/\varepsilon_0 r) \hat{r} \quad \text{(for } a < r < b\text{)}
\]

Much simpler to use \( D \).
Gauss's Law: \[ \nabla \cdot \mathbf{D} = \rho_f \quad \oint \mathbf{D} \cdot d\mathbf{a} = Q_{f_{enc}} \]

Problem 4.16   Suppose the field inside a large piece of dielectric is \( E_0 \), so that the electric displacement is: \( D_0 = \varepsilon_0 E_0 + P \)

Now a small spherical cavity is hollowed out of the material.

Find the field \( E \) at the center of the cavity in terms of \( E_0 \) and \( P \).
Also find the displacement \( D \) at the center of the cavity in terms of \( D_0 \) and \( P \).

Assume the cavities are small enough that \( P \), \( E_0 \), and \( D_0 \) are essentially uniform.

**Hint** ➔ Carving out a cavity is the same as superimposing an object of the same shape but opposite polarization.

\[ E_P = -\frac{P}{3\varepsilon_0} \]

\[ E = E_0 - (E_P) \]

\[ E = E_0 + \frac{1}{3\varepsilon_0}P \]

\[ D = \varepsilon_0 E \]

\[ = \varepsilon_0 E_0 + \frac{1}{3}P \]

\[ = D_0 - P + \frac{1}{3}P \]

\[ D = D_0 - \frac{2}{3}P \]
4.3.2 A Deceptive Parallel: Misleading in comparison between \( \mathbf{E} \) and \( \mathbf{D} \)

\[
\nabla \cdot \mathbf{D} = \rho_f \quad \text{(Let's compare it with Gauss's law)}
\]
\[
\nabla \cdot \mathbf{E} = \rho
\]

\( \rightarrow \) Only the total charge density \( \rho \) is replaced by the free charge density \( \rho_f \). \( \mathbf{D} \) is substituted for \( \varepsilon_0 \mathbf{E} \).

\( \rightarrow \) \( \mathbf{D} \) is "just like" \( \mathbf{E} \) (apart from the factor \( \varepsilon_0 \)), except that its source is \( \rho_f \) instead of \( \rho \).

But, \( \mathbf{D} \) is completely different from \( \mathbf{E} \):

There is no "Coulomb's law" for \( \mathbf{D} \):
\[
\mathbf{D}(\mathbf{r}) \neq \frac{1}{4\pi} \int \frac{\mathbf{r}'}{\varepsilon_0 \rho_f(\mathbf{r}') d\tau'}
\]

In the case of electrostatic fields, the curl of \( \mathbf{E} \) is always zero.
\[
\nabla \times \mathbf{E} = 0
\]

\( \rightarrow \) But the curl of \( \mathbf{D} \) is not always zero.
\[
\nabla \times \mathbf{D} = \varepsilon_0 (\nabla \times \mathbf{E}) + (\nabla \times \mathbf{P}) = \nabla \times \mathbf{P}
\]

\( \rightarrow \) Because \( \nabla \times \mathbf{D} \neq 0 \), \( \mathbf{D} \) cannot be expressed as the gradient of a scalar.

\( \rightarrow \) there is no "potential" for \( \mathbf{D} \).
4.3.3 Boundary Conditions

\[ \mathbf{\nabla} \cdot \mathbf{D} = \rho_f \rightarrow \oint_S D \cdot d\mathbf{a} = (\mathbf{D}_1 \cdot \mathbf{a}_n + \mathbf{D}_2 \cdot \mathbf{a}_n) \Delta S = \mathbf{a}_n \cdot (\mathbf{D}_1 - \mathbf{D}_2) \Delta S = (\mathbf{D}_{1n} - \mathbf{D}_{2n}) \Delta S \]

\[ \rightarrow \int_V \rho_f d\mathbf{v} = (\rho_f \Delta h) \Delta S \]

\[ \mathbf{D}_{1n} - \mathbf{D}_{2n} = \sigma_f \]

\[ \sigma_f = \lim_{\Delta h \to 0} \rho_f \Delta h \]

\[ \nabla \times \mathbf{D} = \nabla \times \mathbf{P} \rightarrow \oint_C D \cdot dl = (\mathbf{D}_1 \cdot l_2 + \mathbf{D}_2 \cdot l_1) = l_2 \cdot (\mathbf{D}_1 - \mathbf{D}_2) = (\mathbf{D}_{1t} - \mathbf{D}_{2t}) \]

\[ \mathbf{D}_{1t} - \mathbf{D}_{2t} = \mathbf{P}_{1t} - \mathbf{P}_{2t} \]

Note for E:

\[ E_{1n} - E_{2n} = \frac{\sigma_{\text{enclosed}}}{\varepsilon_0} = \frac{\sigma_f + \sigma_b}{\varepsilon_0} \]

\[ E_{1t} - E_{2t} = 0 \]

Summary of Boundary conditions:

\[ \mathbf{E}_{\text{above}} \parallel - \mathbf{E}_{\text{below}} \parallel = 0 \]
\[ \mathbf{D}_{\text{above}} \parallel - \mathbf{D}_{\text{below}} \parallel = \mathbf{P}_{\text{above}} \parallel - \mathbf{P}_{\text{below}} \parallel \]

\[ \mathbf{E}_{\text{above}} \perp - \mathbf{E}_{\text{below}} \perp = \frac{1}{\varepsilon_0} \sigma_\parallel \]
\[ \mathbf{D}_{\text{above}} \perp - \mathbf{D}_{\text{below}} \perp = \sigma_f \]

\[ (\sigma = \sigma_f + \sigma_b) \]
4.4 Linear Dielectrics

4.4.1 Susceptibility, Permittivity, Dielectric Constant

\[ P = \varepsilon_0 \chi_e E \]

\textit{In linear dielectrics,} \( P \) is proportional to \( E \), provided \( E \) is not too strong.

\( \chi_e \) : Electric susceptibility \hspace{1cm} (It would be a tensor in general cases)

\[ D = \varepsilon_0 E + P = \varepsilon_0 \left(1 + \chi_e\right) E \]

\[ D = \varepsilon E \]

\[ \varepsilon = \varepsilon_0 \left(1 + \chi_e\right) \] : Permittivity

\[ \varepsilon_r = 1 + \chi_e = \frac{\varepsilon}{\varepsilon_0} \] : Relative permittivity \hspace{1cm} (Dielectric constant)

Note that \( E \) is the total field from free charges and the polarization itself.

\( \Rightarrow \) If, for instance, we put a piece of dielectric into an external field \( E_0 \),

\( \Rightarrow \) we cannot compute \( P \) directly from \( P = \varepsilon_0 \chi_e E; \quad E \neq E_0 \)

\( \Rightarrow \) \( E_0 \) produces \( P \), \( P \) will produce its own field, this in turn modifies \( P \), which ... Breaking where?
Problem 4.41  In a linear dielectric, the polarization is proportional to the field: 

\[ P = \varepsilon_0 \chi_e E. \]

If the material consists of atoms (or nonpolar molecules), the induced dipole moment of each one is likewise proportional to the field 

\[ p = \alpha E. \]

What is the relation between atomic polarizability \( \alpha \) and susceptibility \( \chi_e \)?

Note that, the atomic polarizability \( \alpha \) was defined for an isolated atom subject to an external field coming from somewhere else, 

\[ E_{\text{else}} \rightarrow p = \alpha E_{\text{else}} \]

For \( N \) atoms in unit volume, the polarization can be set 

\[ \rightarrow P = N \alpha E_{\text{else}} \]

There is another electric field, \( E_{\text{self}} \), produced by the polarization \( P \):

\[ E_{\text{self}} = -\frac{1}{4\pi \varepsilon_0} \frac{q d}{R^3} \rightarrow p = q d = \left( \frac{4}{3} \pi R^3 \right) P, \rightarrow E_{\text{self}} = -\frac{1}{3 \varepsilon_0} P \]

\[ \rightarrow E_{\text{self}} = -\frac{N \alpha}{3 \varepsilon_0} E_{\text{else}} \]

Therefore, the total field is

\[ E = E_{\text{self}} + E_{\text{else}} = \left( 1 - \frac{N \alpha}{3 \varepsilon_0} \right) E_{\text{else}} \]

The total field \( E \) finally produce the polarization \( P \):

\[ P = \frac{N \alpha}{(1 - N \alpha/3 \varepsilon_0)} E = \varepsilon_0 \chi_e E \rightarrow \alpha = \frac{\varepsilon_0}{N} \frac{\chi_e}{1 + \chi_e/3} \]  or  \[ \alpha = \frac{3 \varepsilon_0}{N} \left( \frac{\varepsilon_r - 1}{\varepsilon_r + 2} \right) \]

\[ \Rightarrow \text{ Clausius-Mossotti formula} \]
Example 4.5  A metal sphere of radius \(a\) carries a charge \(Q\). It is surrounded, out to radius \(b\), by linear dielectric material of permittivity \(\varepsilon\). Find the potential \(V\) at the center (relative to infinity).

To compute \(V\), we need to know \(E\). \(\rightarrow\) Because it has spherical symmetry, calculate \(D\) first.

Inside the metal sphere, \(E = P = D = 0\).

Outside the metal sphere, \(D = \frac{Q}{4\pi r^2} \hat{r}\), for all points \(r > a\). \(\rightarrow\) \(E = \begin{cases} \frac{Q}{4\pi \varepsilon r^2} \hat{r}, & \text{for } a < r < b, \\ \frac{Q}{4\pi \varepsilon_0 r^2} \hat{r}, & \text{for } r > b. \end{cases}\)

The potential at the center \(\rightarrow\) \(V = -\int_0^1 E \cdot dl = -\int_\infty^b \left(\frac{Q}{4\pi \varepsilon_0 r^2}\right) dr - \int_b^a \left(\frac{Q}{4\pi \varepsilon r^2}\right) dr - \int_0^a (0) dr = \frac{Q}{4\pi} \left(\frac{1}{\varepsilon_0 b} + \frac{1}{\varepsilon a} - \frac{1}{\varepsilon b}\right)\).

Note that the polarization \(P\) in the dielectric is \(P = \varepsilon_0 \chi_e E = \frac{\varepsilon_0 \chi_e Q}{4\pi \varepsilon r^2} \hat{r}\) for \(a < r < b\).

\(\rho_b = -\nabla \cdot P = 0\) \hspace{1cm} \(\sigma_b = P \cdot \hat{n} = \begin{cases} \frac{\varepsilon_0 \chi_e Q}{4\pi \varepsilon b^2}, & \text{at the outer surface}, \\ -\frac{\varepsilon_0 \chi_e Q}{4\pi \varepsilon a^2}, & \text{at the inner surface}. \end{cases}\)
The two vectors of \( E \) and \( D \) is parallel in linear dielectric.

In linear dielectrics, \( P = \varepsilon_0 \chi_e E \longrightarrow D = \varepsilon_0 E + P = \varepsilon_0 (1 + \chi_e) E = \varepsilon E \)

\[
\nabla \times E = 0 \rightarrow \nabla \times D = 0 ?? \text{ since } E \text{ and } D \text{ are parallel} ??
\]

\( \Rightarrow \) Unfortunately, it does not in general:

\[
\nabla \times D = \varepsilon_0 (\nabla \times E) + (\nabla \times P) = \nabla \times P \neq 0
\]

BUT, if the space is entirely filled with a homogeneous \((\chi_e \text{ is the same in all position})\) linear dielectric,

\[
\nabla \cdot D = \rho_f \text{ and } \nabla \times D = 0 \quad \text{Compare} \quad \nabla \cdot E = \frac{1}{\varepsilon_0} \rho \text{ and } \varepsilon_0 \left( \nabla \times E \right) = 0
\]

\[
E = \frac{1}{\varepsilon} D = \frac{1}{\varepsilon_r} E_{\text{vac}}
\]

Conclusion: When all space is filled with a homogeneous linear dielectric, the field everywhere is simply reduced by a factor of one over the dielectric constant.

For example, if a free charge \( q \) is embedded in a large dielectric \( \varepsilon \rightarrow E = \frac{1}{4\pi \varepsilon} \frac{q}{r^2} \hat{r} \)

\( \Rightarrow \text{It becomes weaker!} \) Because \( P \) shield the charge.
4.4.2 Boundary Value Problems with Linear Dielectrics

\[ \rho_b = -\nabla \cdot \mathbf{P} \]

\[ \rho = -\nabla \cdot \mathbf{P} = -\nabla \cdot \left( \epsilon_0 \frac{\chi e}{\epsilon} \mathbf{D} \right) = -\left( \frac{\chi e}{1 + \chi e} \right) \rho_f \]

- The bound charge density is proportional to the free charge density.
- If \( \rho_f = 0; \rho_b = 0 \) \( \rightarrow \) Any net charge must reside at the surface
- Within a dielectric when \( \rho_f = 0 \); the potential obeys Laplace’s equation.

\[ D_{1n} - D_{2n} = \sigma_f \]

\[ \epsilon_{\text{above}} E_{\text{above}} - \epsilon_{\text{below}} E_{\text{below}} = \sigma_f \]

\[ E_{\text{above}} - E_{\text{below}} = \frac{1}{\epsilon_0} \frac{l}{\sigma} \quad \text{(If embedded in vacuum)} \]

OR

\[ \epsilon_{\text{above}} \frac{\partial V_{\text{above}}}{\partial n} - \epsilon_{\text{below}} \frac{\partial V_{\text{below}}}{\partial n} = -\sigma_f \]

\[ V_{\text{above}} - V_{\text{below}} = -\int_a^b \mathbf{E} \cdot d\mathbf{l} \]

\[ V_{\text{above}} = V_{\text{below}} \]
Summary on Boundary Conditions for Electrostatics

Electric field

\[ E_{\text{above}}^\parallel - E_{\text{below}}^\parallel = 0 \]

\[ E_{\text{above}}^\perp - E_{\text{below}}^\perp = \frac{1}{\varepsilon_0} \sigma \quad (\sigma = \sigma_f + \sigma_b) \]

Field displacement

\[ D_{\text{above}}^\parallel - D_{\text{below}}^\parallel = P_{\text{above}}^\parallel - P_{\text{below}}^\parallel \]

\[ D_{\text{above}}^\perp - D_{\text{below}}^\perp = \sigma_f \]

\[ \epsilon_{\text{above}} E_{\text{above}}^\perp - \epsilon_{\text{below}} E_{\text{below}}^\perp = \sigma_f \]

\[ \epsilon_{\text{above}} \frac{\partial V_{\text{above}}}{\partial n} - \epsilon_{\text{below}} \frac{\partial V_{\text{below}}}{\partial n} = -\sigma_f \]

Potential

\[ V_{\text{above}} = V_{\text{below}} \]
Problem 4.23  Find the electric field inside the sphere ($\varepsilon_r$).
Use the following method of **successive approximations**:

→ First pretend the field inside is just $E_0$

→ Write down the resulting polarization $P_0$  

→ $P_0$ generates a field of its own, $E_1$

→ $E_1$ modifies the polarization by an amount $P_1$

→ $P_1 \to E_2$

→ and so on.

→ The resulting field is  

$$E = E_0 + E_1 + E_2 + \cdots = \left[ \sum_{n=0}^{\infty} \left( \frac{-\chi_e}{3} \right)^n \right] E_0$$

$$E = \frac{1}{1 - x} \quad \Rightarrow \quad E = \frac{1}{(1 + \chi_e/3)} E_0$$

$$E = \frac{1}{(\chi_e/3) + 1} E_0 \to E_0 = E + (\chi_e/3) E \quad \Rightarrow \quad P = \varepsilon_0 \chi_e E \to E_0 = E + \frac{1}{3\varepsilon_0} P \quad \Rightarrow \quad E = E_0 - \frac{1}{3\varepsilon_0} P \quad (\text{if } E_0 = 0)$$
Energy in Dielectric Systems

\[ W = \frac{\varepsilon_0}{2} \int \varepsilon_r E^2 \, d\tau = \frac{1}{2} \int \mathbf{D} \cdot \mathbf{E} \, d\tau \]

Let’s prove it.

As we bring in free charge, a bit at a time, the work done on the incrementally free charge \( \Delta \rho_f \) is

\[
\Delta W = \int (\Delta \rho_f) V \, d\tau
\]

\[
= \int [\nabla \cdot (\Delta \mathbf{D})] V \, d\tau \quad \text{since} \quad \nabla \cdot \mathbf{D} = \rho_f
\]

\[
= \int \nabla \cdot [(\Delta \mathbf{D}) V] \, d\tau + \int (\Delta \mathbf{D}) \cdot \mathbf{E} \, d\tau
\]

\[
= \int (\Delta \mathbf{D}) \cdot \mathbf{E} \, d\tau
\]

The divergence theorem turns the first term into a surface integral, which vanishes over all of space.

If the medium is a linear dielectric, \( \mathbf{D} = \varepsilon \mathbf{E} \)

\[
\Delta W = \Delta \left( \frac{1}{2} \int \mathbf{D} \cdot \mathbf{E} \, d\tau \right) \quad \text{Total work done = Stored energy}
\]

\[ W = \frac{1}{2} \int \mathbf{D} \cdot \mathbf{E} \, d\tau \]

Total work done = Stored energy
Forces on Dielectrics

\[ \sigma = \varepsilon_0 E = \varepsilon_0 V/d \]
\[ \sigma_f = \varepsilon_0 \varepsilon_r V/d = \varepsilon_r \sigma \]

\[ C = \frac{Q}{V} = \frac{1}{V} \left[ w x \sigma + w(l-x)\sigma_f \right] = \frac{\varepsilon_0 w}{d} \left[ \varepsilon_r l - (\varepsilon_r - 1)x \right] = \frac{\varepsilon_0 w}{d} \left( \varepsilon_r l - \chi_e x \right) \]

The energy stored in the capacitor with a constant charge \( Q \) is

\[ W = \frac{1}{2} \frac{Q^2}{C} \quad \rightarrow \quad F = -\frac{dW}{dx} = \frac{1}{2} \frac{Q^2}{C^2} \frac{dC}{dx} = \frac{1}{2} V^2 \frac{dC}{dx} \]
\[ \frac{dC}{dx} = -\frac{\varepsilon_0 \chi_e w}{d} \]

\[ F = -\frac{\varepsilon_0 \chi_e w}{2d} V^2 \quad \text{(the dielectric is pulled into the capacitor)} \]