Chapter 5. Magnetostatics

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5.1 The Lorentz Force Law

5.1.1 Magnetic Fields

Consider the forces between charges in motion

Attraction of parallel currents and Repulsion of antiparallel ones:

→ How do you explain this?

→ Does charging up the wires make simply the electrical repulsion of like charges? The wires are in fact electrically neutral!

→ It is not electrostatic in nature.

→ A moving charge generates a Magnetic field $B$.

If you grab the wire with your right hand-thumb in the direction of the current → your fingers curl around in the direction of the magnetic field.

How can such a field lead to a force of attraction on a nearby parallel current?

How do you calculate the magnetic field?
5.1.2 Magnetic Forces \( \mathbf{F}_{\text{mag}} = Q(\mathbf{v} \times \mathbf{B}) \)

In the presence of both *electric and magnetic fields*, the net force on Q would be

\[
\mathbf{F} = Q[\mathbf{E} + (\mathbf{v} \times \mathbf{B})] \quad \Rightarrow \text{Lorentz force law}
\]

\[
\Rightarrow \text{It is a fundamental axiom justified in experiments.}
\]

The *magnetic force* in a charge Q, moving with *velocity* \( \mathbf{v} \) in a *magnetic field* \( \mathbf{B} \), is

\[
\mathbf{F}_{\text{mag}} = Q(\mathbf{v} \times \mathbf{B}) \quad \Rightarrow \text{Magnetic forces, which do not work!}
\]

\[
\Rightarrow \text{For if Q moves an amount } dl = v \, dt, \text{ the work done is}
\]

\[
dW_{\text{mag}} = \mathbf{F}_{\text{mag}} \cdot dl = Q(\mathbf{v} \times \mathbf{B}) \cdot \mathbf{v} \, dt = 0
\]

This follows because \( (\mathbf{v} \times \mathbf{B}) \) is perpendicular to \( \mathbf{v} \), so \( (\mathbf{v} \times \mathbf{B}) \cdot \mathbf{v} = 0 \).

\[
\Rightarrow \text{Magnetic forces may alter the direction in which a particle moves, but they cannot speed it up or slow it down.}
\]

Stationary charges produce electric fields that are constant in time \( \Rightarrow \text{Electrostatics} \)

Steady currents produce magnetic fields that are constant in time \( \Rightarrow \text{Magnetostatics} \)
Lorentz Forces: \[ \mathbf{F} = Q[\mathbf{E} + (\mathbf{v} \times \mathbf{B})] \]

**Example 5.2  Cycloid motion**

If a charge \( Q \) at rest is released from the origin, what path will it follow?

Let's think it through qualitatively, first.

\( \rightarrow \) Initially, the particle is at rest, so the magnetic force is zero.
\( \rightarrow \) The electric field accelerates the charge in the \( z \)-direction.
\( \rightarrow \) As it picks up speed, a magnetic force develops.
\( \rightarrow \) The magnetic field pulls the charge around to the right.
\( \rightarrow \) The faster it goes, the stronger magnetic force becomes; it curves the particle back around towards the \( y \) axis.
\( \rightarrow \) The charge is moving against the electrical force, so it begins to slow down.
\( \rightarrow \) The magnetic force then decreases and the electrical force takes over, bringing the charge to rest at point \( a \).
\( \rightarrow \) There the entire process commences anew, carrying the particle over to point \( b \), and so on.

Now let's do it again quantitatively.

\[ \mathbf{v} = (0, \hat{y}, \hat{z}) \quad \rightarrow \quad \mathbf{v} \times \mathbf{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & \hat{y} & \hat{z} \\ B & 0 & 0 \end{vmatrix} = B\hat{z}\hat{y} - B\hat{y}\hat{z} \]

\[ \mathbf{F} = Q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) = Q(\hat{E}\hat{z} + B\hat{z}\hat{y} - B\hat{y}\hat{z}) = m\mathbf{a} = m(\dot{y}\hat{y} + \dot{z}\hat{z}) \]

\[ \begin{align*}
QB\dot{z} &= m\ddot{y} \\
QE - QB\dot{y} &= m\ddot{z} \\
\end{align*} \quad \text{(cyclo}t\text{ron frequency)}
\]

At \( t = 0 \):
\[ \begin{align*}
y(0) &= z(0) = 0 \\
\dot{y}(0) &= \dot{z}(0) = 0 \\
\end{align*} \rightarrow \begin{align*}
y(t) &= \frac{E}{\omega B} (\omega t - \sin \omega t) \\
z(t) &= \frac{E}{\omega B} (1 - \cos \omega t) \\
\end{align*} \]

At \( z = 0 \):
\[ |F_E| = |F_{mag}| \rightarrow \mathbf{v} = \frac{E}{B} \rightarrow R = \frac{v}{\omega} = \frac{E}{\omega B} \rightarrow \sin^2 \omega t + \cos^2 \omega t = 1 \rightarrow (y - R\omega t)^2 + (z - R)^2 = R^2. \]

This is the formula for a circle, of radius \( R \), whose center \((0, R\omega t, R)\) travels in the \( y \)-direction at a constant speed, \( v = E/B \).

\( \rightarrow \) Cycloid motion
5.1.3 Currents \( \Rightarrow \) \( 1 \text{ A} = 1 \text{ C/s} \) \( \Rightarrow \) Ampere (A)

When a line charge \( \lambda \) traveling down at velocity \( v \) \( \Rightarrow \) \( I = \lambda v \)

The magnetic force on a segment of current-carrying wire is

\[
F_{\text{mag}} = \int (v \times B) \, dq = \int (v \times B) \lambda \, dl = \int (I \times B) \, dl
\]

\( I \) and \( dl \) both point in the same direction, \( \Rightarrow \)

\[
F_{\text{mag}} = I \int (dl \times B)
\]

For a steady current (constant in magnitude) along the wire \( \Rightarrow \)

\[
F_{\text{mag}} = I \int (dl \times B)
\]

When charge flows over a surface with surface current density, \( K \):

\[
K = \frac{dI}{dl_\perp} = \sigma v \quad \Rightarrow \text{current per unit width-perpendicular-to-flow}
\]

\[
F_{\text{mag}} = \int (v \times B) \, dq = \int (v \times B) \sigma \, da \quad \Rightarrow \quad F_{\text{mag}} = \int (K \times B) \, da
\]

When charge flows over a volume with volume current density, \( J \):

\[
J = \frac{dI}{d\alpha_\perp} = \rho v \quad \Rightarrow \text{current per unit area-perpendicular-to-flow}
\]

\[
F_{\text{mag}} = \int (v \times B) \, dq = \int (v \times B) \rho \, d\tau \quad \Rightarrow \quad F_{\text{mag}} = \int (J \times B) \, d\tau
\]
Continuity Equation of current density

\[ \mathbf{J} = \frac{dI}{da_\perp} = \rho \mathbf{v} \]

⇒ The current crossing a surface \( S \) can be written as

\[ I = \int_S J \, da_\perp = \int_S \mathbf{J} \cdot d\mathbf{a} \]

The total charge per unit time leaving a volume \( V \) is

\[ \oint_S \mathbf{J} \cdot d\mathbf{a} = \int_V (\nabla \cdot \mathbf{J}) \, d\tau \]

Because charge is conserved,

\[ \int_V (\nabla \cdot \mathbf{J}) \, d\tau = -\frac{d}{dt} \int_V \rho \, d\tau = -\int_V \left( \frac{\partial \rho}{\partial t} \right) \, d\tau \]

\[ \nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} \quad \Rightarrow \text{Continuity equation (Local charge conservation)} \]

For a steady current,

\[ \frac{\partial \rho}{\partial t} = 0 \quad \Rightarrow \quad \nabla \cdot \mathbf{J} = 0 \quad \oint_S \mathbf{J} \cdot ds = 0 \rightarrow \sum_{j=0} I_j = 0 \quad (\text{Kirchhoff’s current law}) \]
5.2 The Biot-Savart Law

5.2.1 Steady Currents

Steady current ➔ A continuous flow that has been going on forever, without change and without charge piling up anywhere.

For a steady current, constant magnetic fields: Magnetostatics ➔ \( \nabla \cdot J = 0 \)

5.2.2 The Magnetic Field of a Steady Current

\[
B(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{I} \times \mathbf{\hat{r}}}{r^2} \, dl' = \frac{\mu_0}{4\pi} I \int \frac{\mathbf{d}l' \times \mathbf{\hat{r}}}{r^2}
\]

➔ Biot-Savart law

(Permeability of free space) \( \mu_0 = 4\pi \times 10^{-7} \text{ N/A}^2 \)

(Unit of B, Tesla) 1 T = 1 N/(A \cdot m)

For surface and volume currents

\[
B(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{K}(\mathbf{r'}) \times \mathbf{\hat{r}}}{r^2} \, d\mathbf{a'}
\]

\[
B(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r'}) \times \mathbf{\hat{r}}}{r^2} \, d\mathbf{\tau'}
\]

➔ Biot-Savart law

**Biot-Savart law in magnetostatics plays a role analogous to Coulomb's law in electrostatics**

\(1/r^2\) dependence is common
The Biot-Savart Law \[ \mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{I} \times \hat{z}}{r^2} \, dl' = \frac{\mu_0}{4\pi} \int \frac{dl' \times \hat{z}}{r^2}. \]

**Example 5.5** Find the magnetic field a distance \( s \) from a long straight wire carrying a steady current \( I \).

\((dl' \times \hat{z})\) points out of the page, and has the magnitude of

\[ dl' \sin\alpha = dl' \cos\theta \quad I' = s \tan\theta \rightarrow dl' = \frac{s}{\cos^2\theta} \, d\theta \quad s = z \cos\theta \rightarrow \frac{1}{r^2} = \frac{\cos^2\theta}{s^2} \]

\[ B = \frac{\mu_0 I}{4\pi} \int_{\theta_1}^{\theta_2} \left( \frac{\cos^2\theta}{s^2} \right) \left( \frac{s}{\cos^2\theta} \right) \cos\theta \, d\theta = \frac{\mu_0 I}{4\pi s} \int_{\theta_1}^{\theta_2} \cos\theta \, d\theta = \frac{\mu_0 I}{4\pi s} (\sin\theta_2 - \sin\theta_1). \]

In the case of an infinite wire, \( \theta_1 = -\pi/2 \) and \( \theta_2 = \pi/2 \), \[ B = \frac{\mu_0 I}{2\pi s}. \]

As an application, let's find the force of attraction between two wires carrying currents, \( I_1 \) and \( I_2 \).

The field at (2) due to (1) is \( B = \frac{\mu_0 I_1}{2\pi d} \rightarrow \) It points into the page.

\[ \mathbf{F}_{\text{mag}} = I \int (dl \times \mathbf{B}) \rightarrow \mathbf{F} = I_2 \left( \frac{\mu_0 I_1}{2\pi d} \right) \int dl \rightarrow \text{It is directed towards (1)} \]

The force per unit length \( f = \frac{\mu_0 I_1 I_2}{2\pi d} \rightarrow \text{It is attractive to each other}. \]
5.3 The Divergence and Curl of B: \[ \nabla \cdot \mathbf{B} \quad \nabla \times \mathbf{B} \]

5.3.1 Straight-Line Currents

The magnetic field of an infinite straight wire looks to have a *nonzero curl*. Let's calculate the Curl of \( \mathbf{B} \).

If we use cylindrical coordinates \((s, \phi, z)\), with the current flowing along the \(z\) axis,

\[
\mathbf{B} = \frac{\mu_0 I}{2\pi s} \hat{\phi} \quad d\mathbf{l} = ds \hat{s} + s d\phi \hat{\phi} + dz \hat{z},
\]

\(\leftrightarrow\) We knew it for the infinite long wire from Biot-Savart law)

\[
\int \mathbf{B} \cdot d\mathbf{l} = \frac{\mu_0 I}{2\pi} \int \frac{1}{s} s d\phi = \frac{\mu_0 I}{2\pi} \int_0^{2\pi} d\phi = \mu_0 I \quad \leftrightarrow \text{It is independent of } s.
\]

Now suppose we have a *bundle* of straight wires:

\[
\int \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}} \quad I_{\text{enc}} = \int \mathbf{J} \cdot d\mathbf{a}
\]

Applying Stokes' theorem \(\Rightarrow\)

\[
\int (\nabla \times \mathbf{B}) \cdot d\mathbf{a} = \mu_0 \int \mathbf{J} \cdot d\mathbf{a}
\]

\[\nabla \times \mathbf{B} = \mu_0 \mathbf{J}\]

\(\Rightarrow\) But, this derivation is seriously flawed by the restriction to infinite straight line currents. Let's use the Biot-Savart Law for general derivation.
The Divergence of $\mathbf{B}$: $\nabla \cdot \mathbf{B} = 0$

The Biot-Savart law for the general case of a volume current is

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}') \times \hat{\mathbf{z}}}{r^2} \, d\tau'$$

$\mathbf{B}$ is a function of $(x, y, z)$

$\mathbf{J}$ is a function of $(x', y', z')$

$\mathbf{r} = (x - x')\hat{\mathbf{x}} + (y - y')\hat{\mathbf{y}} + (z - z')\hat{\mathbf{z}}$

$d\tau' = dx' \, dy' \, dz'$

Applying the divergence with respect to the unprimed coordinates:

$$\nabla \cdot \mathbf{B} = \frac{\mu_0}{4\pi} \int \nabla \cdot \left( \mathbf{J} \times \frac{\hat{\mathbf{z}}}{r^2} \right) \, d\tau'$$

$$\nabla \cdot \left( \mathbf{J} \times \frac{\hat{\mathbf{z}}}{r^2} \right) = \frac{\hat{\mathbf{z}}}{r^2} \cdot (\nabla \times \mathbf{J}) - \mathbf{J} \cdot \left( \nabla \times \frac{\hat{\mathbf{z}}}{r^2} \right)$$

$\nabla \times \mathbf{J} = 0$, because $\mathbf{J}$ doesn't depend on the unprimed variables $(x, y, z)$

$\nabla \times (\hat{\mathbf{z}}/r^2) = 0$
The Curl of B: \( \nabla \times \mathbf{B} = \mu_0 \mathbf{J} \)

\[
\mathbf{B}(r) = \frac{\mu_0}{4\pi} \int \mathbf{J}(r') \times \frac{\hat{r}}{r'^2} \, d\tau'
\]

\( \mathbf{B} \) is a function of \((x, y, z)\)

\( \mathbf{J} \) is a function of \((x', y', z')\)

\[
\mathbf{r} = (x - x') \hat{x} + (y - y') \hat{y} + (z - z') \hat{z}
\]

\[
d\tau' = dx' \, dy' \, dz'
\]

**Applying the curl** with respect to the *unprimed* coordinates:

\[
\nabla \times \mathbf{B} = \frac{\mu_0}{4\pi} \int \nabla \times \left( \mathbf{J} \times \frac{\hat{r}}{r'^2} \right) \, d\tau'
\]

\( \nabla \cdot \left( \frac{\hat{r}}{r'^2} \right) = 4\pi \delta^3(\mathbf{r}) \)

\[
\mathbf{J}(\mathbf{r}') 4\pi \delta^3(\mathbf{r} - \mathbf{r}') \, d\tau' = \mu_0 \mathbf{J}(\mathbf{r})
\]

The \( x \) component, in particular, is

\[
\left( \mathbf{J} \cdot \nabla' \right) \left( \frac{x - x'}{r'^3} \right) = \nabla' \left( \frac{x - x'}{r'^3} \right) \mathbf{J} - \left( \frac{x - x'}{r'^3} \right) (\nabla' \cdot \mathbf{J})
\]

For *steady* currents the divergence of \( \mathbf{J} \) is zero

\[
\int \nabla' \left( \frac{x - x'}{r'^3} \right) \mathbf{J} \, d\tau' = \oint_S \frac{x - x'}{r'^3} \mathbf{J} \cdot d\mathbf{a}' = 0
\]

We can have the *boundary surface large enough*, so the current is zero on the surface (all current is safely inside)
5.3.3 Applications of Ampere's Law

\[ \nabla \times B = \mu_0 J \]

\[ \int (\nabla \times B) \cdot da = \oint B \cdot dl = \mu_0 \int J \cdot da \]

Which direction through the surface corresponds to a positive current?

\[ \Rightarrow \text{Keep the Right-Hand Rule.} \]

\[ \Rightarrow \text{If the fingers of your right hand indicate the direction of integration around the boundary, then your thumb defines the direction of a positive current.} \]

Electrostatics: Coulomb law \[\Rightarrow\] Gauss’s law

Magnetostatics: Biot-Savart law \[\Rightarrow\] Ampere’s law

Like Gauss’s law, Ampere’s law is useful only when the symmetry of the problem enables you to pull B outside the integral.
Ampere's Law

\[ \oint B \cdot dl = \mu_0 I_{enc} \]

Example 5.7  Find the magnetic field a distance \( s \) from a long straight wire.

\[ \oint B \cdot dl = B \oint dl = B2\pi s = \mu_0 I_{enc} = \mu_0 I \quad \rightarrow \quad B = \frac{\mu_0 I}{2\pi s} \]

Example 5.8  Find the magnetic field of an infinite uniform surface current \( K \), flowing over the xy plane.

\( B \) can only have a \( y \)-component:

\[ \oint B \cdot dl = 2Bl = \mu_0 I_{enc} = \mu_0 Kl \]

\[ \rightarrow \quad B = \begin{cases} +\left(\frac{\mu_0}{2}\right)K\hat{y} & \text{for } z < 0 \\ -\left(\frac{\mu_0}{2}\right)K\hat{y} & \text{for } z > 0 \end{cases} \]

Example 5.9  Find the magnetic field of a very long solenoid, consisting of \( n \) closely wound turns per unit length on a cylinder of radius \( R \) and carrying a steady current \( I \).

\( B \) of an infinite, closely wound solenoid runs parallel to the axis:

\[ \oint B \cdot dl = BL = \mu_0 I_{enc} = \mu_0 nIL \]

\[ \rightarrow \quad B = \begin{cases} \mu_0 nI \hat{z}, & \text{inside the solenoid,} \\ 0, & \text{outside the solenoid} \end{cases} \]
5.3.4 Comparison of Magnetostatics and Electrostatics

The divergence and curl of the electrostatic field are

\[
\begin{align*}
\nabla \cdot \mathbf{E} &= \frac{1}{\varepsilon_0} \rho, \quad \text{(Gauss’s law)}; \\
\nabla \times \mathbf{E} &= 0, \quad \text{(no name)}.
\end{align*}
\]

The divergence and curl of the magnetostatic field are

\[
\begin{align*}
\nabla \cdot \mathbf{B} &= 0, \quad \text{(no name)}; \\
\nabla \times \mathbf{B} &= \mu_0 \mathbf{J}, \quad \text{(Ampère’s law)}
\end{align*}
\]

➔ These 4 equations are Maxwell’s Equations for static electromagnetics

The force law is \( \mathbf{F} = Q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \)

➔ Typically, electric forces are enormously larger than magnetic ones.
5.4 Magnetic Vector Potential

5.1.1 The Vector Potential

In electrostatics, \( \nabla \times \mathbf{E} = 0 \rightarrow \mathbf{E} = -\nabla \phi \rightarrow \) Scalar potential (V)

In magnetostatics, \( \nabla \cdot \mathbf{B} = 0 \rightarrow \mathbf{B} = \nabla \times \mathbf{A} \rightarrow \) Vector potential (A)

(Note) The name is “potential”, but \( \mathbf{A} \) cannot be interpreted as potential energy per unit charge.

So far, for a steady current where \( \nabla \cdot \mathbf{J} = 0 \) the Magnetic field is defined by the Biot-Savart law:

\[
\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}') \times \hat{z}}{z^2} \, d\tau'
\]

This Magnetic field given by the Biot-Savart law always satisfies the relations of

\( \nabla \cdot \mathbf{B} = 0 \) (always zero, even for not steady current)

\( \nabla \times \mathbf{B} = \mu_0 \mathbf{J} \) (as long as the current is steady, \( \nabla \cdot \mathbf{J} = 0 \))

(If the current is not steady, \( \nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} \neq 0 \rightarrow \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 (\frac{\partial \mathbf{D}}{\partial t}) \))

The relations of \( \nabla \times \mathbf{B} = \mu_0 \mathbf{J} \), called Ampere’s Law, is very useful, just like the Gauss’s Law in E.

The relations of \( \nabla \cdot \mathbf{B} = 0 \) is always satisfied for any current flow, therefore we may put the \( \mathbf{B} \) field, either

\( \mathbf{B} = \nabla \times \mathbf{A} \) (for any arbitrary vector \( \mathbf{A} \))

Or, \( \mathbf{B} = \nabla \times (\mathbf{A} + \nabla \lambda) \) (for any arbitrary vector \( \mathbf{A} \) and any scalar \( \lambda \), since \( \nabla \times \nabla \lambda = 0 \))
\[ \mathbf{B} = \nabla \times \mathbf{A} \quad \text{WHY not?} \quad \mathbf{B} = \nabla \times (\mathbf{A} + \nabla \lambda) \]

Actually we want to use the Ampere’s Law, \( \nabla \times \mathbf{B} = \mu_0 \mathbf{J} \), to find out \( \mathbf{B} \) field.

(1) Let’s use the form of \( \mathbf{B} = \nabla \times \mathbf{A} \)

\[ \nabla \times \mathbf{B} = \nabla \times \nabla \times \mathbf{A} = \nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J} \]

If we choose \( \mathbf{A} \) so as to eliminate the divergence of \( \mathbf{A} \): \( \nabla \cdot \mathbf{A} = 0 \)

\[ \nabla \times \mathbf{B} = \mu_0 \mathbf{J} \quad \nabla^2 \mathbf{A} = -\mu_0 \mathbf{J} \quad \text{This is a Poisson’s equation.} \]

We know how to solve it, just like the electrostatic potential problems.

\[ \nabla^2 V = -\frac{\rho}{\varepsilon_0} \quad \text{If \( \rho \) goes to zero at infinite,} \quad V = \frac{1}{4\pi \varepsilon_0} \int \frac{\rho}{r} \, d\tau' \]

\text{Ampere’s Law} \quad \nabla^2 \mathbf{A} = -\mu_0 \mathbf{J} \quad \text{If \( \mathbf{J} \) goes to zero at infinite,} \quad \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}')}{r} \, d\tau' \]

\[ \mathbf{A} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{I}}{r} \, dl' \quad \mathbf{A} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{K}}{r} \, d\alpha' \]

(If the current does not go to zero at infinite, we have to find other ways to get \( \mathbf{A} \).)

\[ \rightarrow \text{In summary, the definition} \quad \mathbf{B} = \nabla \times \mathbf{A} \quad \text{under the condition of} \quad \nabla \cdot \mathbf{A} = 0 \]

\( \rightarrow \) make it possible to transform the Ampere’s law into a Poisson’s equation on \( \mathbf{A} \) and \( \mathbf{J} \)!
\( \mathbf{B} = \nabla \times \mathbf{A} \)  

**WHY not?**  
\( \mathbf{B} = \nabla \times (\mathbf{A} + \nabla \lambda) \)

**2) Now consider the other form,**  
\( \mathbf{B} = \nabla \times (\mathbf{A} + \nabla \lambda) \)

Suppose that a vector potential \( \mathbf{A}_0 \) satisfying \( \mathbf{B} = \nabla \times \mathbf{A}_0 \), but it is not divergenceless, \( \nabla \cdot \mathbf{A}_0 \neq 0 \).

If we add to \( \mathbf{A}_0 \) the gradient of any scalar \( \lambda \), \( \mathbf{A} = \mathbf{A}_0 + \nabla \lambda \), \( \mathbf{A} \) is also satisfied \( \mathbf{B} = \nabla \times \mathbf{A} \).

\[
(\nabla \times (\mathbf{A}_0 + \nabla \lambda) = \nabla \times \mathbf{A} = \mathbf{B} \text{ since } \nabla \times \nabla \lambda = 0)
\]

The divergence of \( \mathbf{A} \) is \( \nabla \cdot \mathbf{A} = \nabla \cdot \mathbf{A}_0 + \nabla^2 \lambda \)

\( \rightarrow \) We can accommodate the condition of \( \nabla \cdot \mathbf{A} = 0 \)  
as long as a scalar function \( \lambda \) can be found that satisfies, \( \nabla^2 \lambda = -\nabla \cdot \mathbf{A}_0 \)

\( \rightarrow \) But this is mathematically identical to Poisson's equation, thus

\[
\lambda = \frac{1}{4\pi} \int \frac{\nabla \cdot \mathbf{A}_0}{r} \, d\tau' \rightarrow \text{ We can always find } \lambda \text{ to make } \nabla \cdot \mathbf{A} = 0
\]

\( \rightarrow \) **Therefore, let's use the simple form of**  
\( \mathbf{B} = \nabla \times \mathbf{A} \)  
**with the divergenceless vector potential,**  
\( \nabla \cdot \mathbf{A} = 0 \)
May we use a scalar potential, \( \mathbf{B} = -\nabla U \), just like used in \( \mathbf{E} = -\nabla V \)?

The introduction of Vector Potential (A) may not be as useful as \( V \),
\( \Rightarrow \) It is still a vector with many components.

\textit{It would be nice if we could get away with a scalar potential}, for instant, \( \mathbf{B} = -\nabla U \)

\( \Rightarrow \) But, this is \textit{incompatible with Ampere's law}, \( \nabla \times \mathbf{B} = \mu_0 \mathbf{J} \)
since the curl of a gradient is always zero \( \Rightarrow \) \( \nabla \times \mathbf{B} = -\nabla \times \nabla U = 0 \)

\( \Rightarrow \) Such a magnetostatic scalar potential \textit{could} be used,
if you stick scrupulously to simply-connected current-free regions (\( \mathbf{J} = 0 \)).

\textbf{Prob. 5-29} Suppose that \( \mathbf{B} = -\nabla U \).
Show, by applying Ampere's law to a path that starts at \( \mathbf{a} \) and circles the wire returning to \( \mathbf{b} \)
that the scalar potential cannot be single-valued, \( U(\mathbf{a}) \neq U(\mathbf{b}) \), even if they are the same point (\( \mathbf{a} = \mathbf{b} \)).

\[
\mu_0 I = \oint \mathbf{B} \cdot d\mathbf{l} = -\int_{\mathbf{a}}^{\mathbf{b}} \nabla U \cdot d\mathbf{l} = -[U(\mathbf{b}) - U(\mathbf{a})]
\]
(by the gradient theorem).

\[ \Rightarrow U(\mathbf{b}) \neq U(\mathbf{a}) \]
5.4.2 Summary; Relations of B – J – A

From just two experimental observations:

1. The principle of superposition - a broad general rule
2. Biot-Savert’s law - the fundamental law of magnetostatics.

\[ A(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{J(r')}{r} \, d\tau' \]

\[ \nabla^2 A = -\mu_0 J \]

\[ B = \nabla \times A; \quad \nabla \cdot B = 0 \]

\[ \oint A \cdot d\mathbf{l} = \int (\nabla \times A) \cdot d\mathbf{a} = \int B \cdot d\mathbf{a} \]

\[ B(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{J(r') \times \hat{z}}{r^2} \, d\tau' \]

\[ \nabla \cdot B = 0 \]

\[ \nabla \times B = \mu_0 J \]

Stoke’s theorem
From just two experimental observations:
1. **the principle of superposition** - a broad general rule
2. **Coulomb’s law** - the fundamental law of electrostatics.

\[ E = \frac{1}{4\pi \varepsilon_0} \int \frac{\rho(\mathbf{r}')}{r^2} \, d\tau' \]

\[ V = \frac{1}{4\pi \varepsilon_0} \int \mathbf{E} \cdot d\mathbf{l} \]

\[ \nabla^2 V = -\frac{\rho}{\varepsilon_0} \]

\[ \mathbf{E} = \frac{1}{4\pi \varepsilon_0} \int \frac{\rho(\mathbf{r}')}{r^2} \, d\tau' \]

\[ \nabla \mathbf{E} = \frac{\rho}{\varepsilon_0} \mathbf{V} \times \mathbf{E} = 0 \]

\[ \nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \]

\[ \nabla \cdot \mathbf{V} = 0 \]

\[ \nabla \times \mathbf{E} = 0 \]

\[ \nabla \times \mathbf{V} = -\frac{\rho}{\varepsilon_0} \]

\[ \mathbf{J} = \frac{\mu_0}{4\pi} \int \mathbf{J}(\mathbf{r}') \times \mathbf{r} \, d\tau' \]

\[ \nabla \times \mathbf{A} = -\mu_0 \mathbf{J} \]

\[ \mathbf{B} = \mu_0 \int \mathbf{J}(\mathbf{r}') \times \mathbf{r} \, d\tau' \]

\[ \nabla \times \mathbf{B} = \mu_0 \mathbf{J} \cdot \mathbf{V} \cdot \mathbf{B} = 0 \]

\[ \nabla \cdot \mathbf{B} = 0 \]

\[ \nabla \cdot \mathbf{A} = 0 \]

\[ \oint A \cdot d\mathbf{l} = \int (\nabla \times A) \cdot d\mathbf{a} = \int B \cdot d\mathbf{a} \]

**Ampere’s Law**

**Stoke’s theorem**
5.4.2 Magnetostatic Boundary Conditions

(Normal components)  \( \nabla \cdot \mathbf{B} = 0 \)

\[ \nabla \cdot \mathbf{B} = 0 \quad \Rightarrow \quad \oint \mathbf{B} \cdot d\mathbf{a} = 0 \quad \Rightarrow \quad B_{\text{above}}^\perp = B_{\text{below}}^\perp \]

(Tangential components)  \( \nabla \times \mathbf{B} = \mu_0 \mathbf{J} \quad \oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}} \)

For an amperian loop running perpendicular to the current,

\[ \oint \mathbf{B} \cdot d\mathbf{l} = (B_{\text{above}}^\parallel - B_{\text{below}}^\parallel)l = \mu_0 I_{\text{enc}} = \mu_0 Kl \quad \Rightarrow \quad B_{\text{above}}^\parallel - B_{\text{below}}^\parallel = \mu_0 K \]

For an amperian loop running parallel to the current,

\[ \oint \mathbf{B} \cdot d\mathbf{l} = (B_{\text{above}}^\parallel - B_{\text{below}}^\parallel)l = \mu_0 I_{\text{enc}} = 0 \quad \Rightarrow \quad B_{\text{above}}^\parallel = B_{\text{below}}^\parallel \]

Finally, we can summarize in a single formula:

\[ B_{\text{above}} - B_{\text{below}} = \mu_0 (\mathbf{K} \times \hat{\mathbf{n}}) \]

The vector potential is continuous across any boundary:

\[ A_{\text{above}} = A_{\text{below}} \]

(Because, \( \nabla \cdot \mathbf{A} = 0 \) guarantees the normal continuity; \( \nabla \times \mathbf{A} = \mathbf{B} \) the tangential one.)

\[ \nabla \times \mathbf{A} = \mathbf{B} \rightarrow \oint \mathbf{A} \cdot d\mathbf{l} = \int \mathbf{B} \cdot d\mathbf{a} = \Phi = 0 \] (the flux through an amperian loop of vanishing thickness is zero)
Summary of Boundary conditions:  \( \mathbf{E} \leftrightarrow \mathbf{B} \)

\[
\begin{align*}
\mathbf{E}_\text{above}^\parallel - \mathbf{E}_\text{below}^\parallel &= 0 \\
\mathbf{E}_\text{above}^\perp - \mathbf{E}_\text{below}^\perp &= \frac{1}{\varepsilon_0} \sigma \\
V_\text{above} &= V_\text{below}
\end{align*}
\]

\[
\begin{align*}
\mathbf{B}_\text{above}^\parallel - \mathbf{B}_\text{below}^\parallel &= \mu_0 K \\
\mathbf{B}_\text{above}^\perp - \mathbf{B}_\text{below}^\perp &= \mu_0 \left( \mathbf{K} \times \mathbf{n} \right) \\
\mathbf{A}_\text{above} &= \mathbf{A}_\text{below}
\end{align*}
\]

\[
\begin{align*}
\frac{\partial \mathbf{A}_\text{above}}{\partial n} - \frac{\partial \mathbf{A}_\text{below}}{\partial n} &= -\mu_0 K
\end{align*}
\]
5.4.3 Multipole Expansion of the Vector Potential

If we want an approximate formula for the vector potential of a localized current distribution, valid at distant points, a multipole expansion is required.

\[ \mathbf{A} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{I}}{r} \, dl' \]

Multipole expansion means to write the potential in the form of a power series in \(1/r\), if \(r\) is sufficiently large.

\[ \frac{1}{r} = \frac{1}{\sqrt{r^2 + (r')^2 - 2rr' \cos \theta'}} = \frac{1}{r} \sum_{n=0}^{\infty} \left( \frac{r'}{r} \right)^n P_n(\cos \theta') \quad (\text{Eq. } 3.94) \]

\[ \mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \oint \frac{1}{r} \, d\mathbf{l}' = \frac{\mu_0 I}{4\pi} \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \oint (r')^n P_n(\cos \theta') \, d\mathbf{l}' \]

\[ \mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \left[ \frac{1}{r} \oint d\mathbf{l}' + \frac{1}{r^2} \int r' \cos \theta' \, d\mathbf{l}' + \frac{1}{r^3} \int (r')^2 \left( \frac{3}{2} \cos^2 \theta' - \frac{1}{2} \right) \, d\mathbf{l}' + \ldots \right] \]

monopole  dipole  quadrupole

Just for comparison:

\[ V(\mathbf{r}) = \frac{1}{4\pi \epsilon_0} \left[ \frac{1}{r} \int \rho(\mathbf{r}') \, d\tau' + \frac{1}{r^2} \int r' \cos \theta' \rho(\mathbf{r}') \, d\tau' + \frac{1}{r^3} \int (r')^2 \left( \frac{3}{2} \cos^2 \theta' - \frac{1}{2} \right) \rho(\mathbf{r}') \, d\tau' + \ldots \right] \]

monopole  dipole  quadrupole
Multipole Expansion of the Vector Potential

\[ A(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \left[ \frac{1}{r} \oint d\mathbf{V} + \frac{1}{r^2} \oint r' \cos \theta' \, d\mathbf{V} + \frac{1}{r^3} \oint (r')^2 \left( \frac{3}{2} \cos^2 \theta' - \frac{1}{2} \right) \, d\mathbf{V} + \ldots \right] \]

- **monopole**
- **dipole**
- **quadrupole**

*(Magnetic Monopole Term) ➞ Always zero!*

- For the integral is just the total displacement around a closed path, \( \oint d\mathbf{V} = 0 \)
- This reflects the fact that there are (apparently) no magnetic monopoles in nature.

*(Magnetic Dipole Term) ➞ It is dominant!*

\[ A_{\text{dip}}(\mathbf{r}) = \frac{\mu_0 I}{4\pi r^2} \oint r' \cos \theta' \, d\mathbf{V} = \frac{\mu_0 I}{4\pi r^2} \oint (\hat{\mathbf{r}} \cdot \mathbf{r}') \, d\mathbf{V} \]

\[ \oint (\hat{\mathbf{r}} \cdot \mathbf{r}') \, d\mathbf{V} = -\hat{\mathbf{r}} \times \oint d\mathbf{a}' \]

\[ \mathbf{m} = I \oint d\mathbf{a} = I \mathbf{a} \quad ➞ \text{Magnetic dipole moment} \]

\[ A_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \hat{\mathbf{r}}}{r^2} \]

\[ p = \oint r' \rho(\mathbf{r}') \, d\mathbf{\tau} \]

\[ V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi \varepsilon_0} \frac{p \cdot \hat{\mathbf{r}}}{r^2} \]
Magnetic Dipole field \[ \mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \hat{\mathbf{r}}}{r^2} \]

\[ \mathbf{m} \equiv I \int d\mathbf{a} = I \mathbf{a} \]

\( \mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0 m \sin \theta}{4\pi r^2} \hat{\phi} \)

\( \mathbf{B}_{\text{dip}}(\mathbf{r}) = \nabla \times \mathbf{A} = \frac{\mu_0 m}{4\pi r^3} (2 \cos \theta \hat{\mathbf{r}} + \sin \theta \hat{\theta}) \)

Field B of a "pure" dipole

Field B of a "physical" dipole
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